Linear System Decomposition into FOT-Deterministic and Residual Components

Antonio Napolitano

Department of Engineering
University of Napoli "Parthenope"
Napoli, Italy

Email: antonio.napolitano@uniparthenope.it antnapol@gmail.com
URL: http://sites.google.com/site/antnapol

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Outline

- Fraction-of-time (FOT) probability: Almost-periodic distribution function and expectation operator
- FOT probability: Deterministic and random signals
- Spectral representation of finite average-power signals
- Linear time-variant systems
- FOT-deterministic linear systems
- Linear system decomposition into deterministic and residual components
- Conclusion
- Bibliography

Almost-Periodic Component Extraction Operator

Let $\mathscr{A}$ be the class of functions $x(t)$ such that the Fourier coefficient

$$x^{\alpha} \triangleq \lim_{T\to\infty} \frac{1}{T} \int_{t_0-T/2}^{t_0+T/2} x(t)\, e^{-j2\pi\alpha t} \mathrm{d}t \quad \text{exists} \;\; \forall\alpha \in \mathbb{R}$$

For every function $x(t) \in \mathscr{A}$, the ***almost-periodic component extraction operator*** $\mathrm{E}^{\{\alpha\}}\{\cdot\}$ extracts all the finite-strength additive sine-wave components of its argument

$$\mathrm{E}^{\{\alpha\}}\{x(t)\} = \sum_{\alpha\in A_x} x^{\alpha}\, e^{j2\pi\alpha t}$$

A_x = countable set

Almost-Periodic Component Extraction Operator (cont'd)

Signal Decomposition (Gardner 1987) (Gardner Brown 1991)

Every function $x(t)$ of the class $\mathscr{A}$ can be decomposed into its almost-periodic component and a residual term not containing any finite-strength additive sine-wave component

$$x(t) = \mathrm{E}^{\{\alpha\}}\{x(t)\} + x_r(t)$$
$$= \sum_{\alpha \in A_x} x^{\alpha}\, e^{j2\pi\alpha t} + x_r(t)$$

with

$$\lim_{T\to\infty} \frac{1}{T} \int_{t_0-T/2}^{t_0+T/2} x_r(t)\, e^{-j2\pi\alpha t}\, \mathrm{d}t = 0 \qquad \forall \alpha \in \mathbb{R}$$

Almost-Periodic Distribution Function and Expectation Operator

Distribution Function (Gardner 1987) (Gardner Brown 1991)

The function

$$F_{x(t)}^{\{\alpha\}}(\xi) \triangleq \mathrm{E}^{\{\alpha\}}\left\{\mathbf{1}_{\{x(t) \leqslant \xi\}}\right\}$$

- as a function of ξ, for any t, is a valid cumulative *distribution function* except for the right-continuity property (in the discontinuity points)
- as a function of t, for any ξ, is almost periodic

Fundamental Theorem of Temporal Expectation (Gardner 1987) (Gardner Brown 1991)

$$\mathrm{E}^{\{\alpha\}}\{g(x(t))\} = \int_{\mathbb{R}} g(\xi)\, \mathrm{d}F_{x(t)}^{\{\alpha\}}(\xi)$$

The almost-periodic component extraction operator $\mathrm{E}^{\{\alpha\}}\{\cdot\}$ is the ***expectation operator*** of the distribution function $F_{x(t)}^{\{\alpha\}}(\xi)$ for the signal $x(t)$.

Deterministic and Random Signals

$$x(t) \text{ almost-periodic function} \quad \Leftrightarrow \quad x(t) \equiv \mathrm{E}^{\{\alpha\}}\{x(t)\}$$

The *almost-periodic functions* are the *deterministic* signals in the almost-periodically time-variant FOT approach. All other signals are the *random* signals.

Spectral Representation

Spectral Representation for Finite Average-Power signals (Wiener 1930), (Doob 1949)

$$x(t) = \int_{\mathbb{R}} e^{j2\pi ft}\, \mathrm{d}Z_x(f)$$

$Z_x(f)$ = *integrated spectrum* of $x(t)$

The above equality means

$$x(t) = \lim_{\Delta f \to 0} \operatorname*{l.i.m.}_{B \to \infty} \int_{-B/2}^{B/2} e^{j2\pi ft}\, \frac{Z_x(f+\Delta f/2) - Z_x(f-\Delta f/2)}{\Delta f}\, \mathrm{d}f$$

$$\lim_{\Delta f \to 0} \left[\lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \left| x(t) - \lim_{B \to \infty} \int_{-B/2}^{B/2} e^{j2\pi ft}\, \frac{Z_x(f+\Delta f/2) - Z_x(f-\Delta f/2)}{\Delta f}\, \mathrm{d}f \right|^2 \mathrm{d}t \right] = 0$$

Spectral Representation (cont'd)

The Fourier transform $X(f)$ of the finite-power signal $x(t)$ can be *formally* defined by

$$\mathrm{d}Z_x(f) = X(f)\,\mathrm{d}f$$

and we can *formally* write

$$x(t) = \int_{\mathbb{R}} X(f)\, e^{j2\pi f t}\,\mathrm{d}f$$

$$X(f) = \int_{\mathbb{R}} x(t)\, e^{-j2\pi f t}\,\mathrm{d}t$$

Linear Time-Variant Systems

Input/Output Relationship in the Time Domain

$$y(t) = \int_{\mathbb{R}} h(t,u)\, x(u)\, \mathrm{d}u$$

Input/Output Relationship in the Frequency Domain

$$\begin{aligned} Y(f) &\triangleq \int_{\mathbb{R}} y(t)\, e^{-j2\pi f t}\, \mathrm{d}t \\ &= \int_{\mathbb{R}} H(f,\lambda)\, X(\lambda)\, \mathrm{d}\lambda \end{aligned}$$

The *transmission function* $H(f,\lambda)$ is the double Fourier transform of the *impulse-response function* $h(t,u)$:

$$H(f,\lambda) \triangleq \int_{\mathbb{R}^2} h(t,u)\, e^{-j2\pi(ft-\lambda u)}\, \mathrm{d}t\, \mathrm{d}u$$

FOT-Deterministic Systems

Definition (Izzo Napolitano 2002): An *FOT-deterministic system* is a possibly complex (and not necessarily linear) system that for every deterministic (i.e., almost periodic) input signal delivers a deterministic output signal.

$$x_\lambda(t) = e^{j2\pi\lambda t} \;\Rightarrow\; y_\lambda(t) = \sum_{\sigma \in \mathbb{S}} G_\sigma(\lambda)\, e^{j2\pi\varphi_\sigma(\lambda)t}$$

$\mathbb{S}$ = countable set

$\varphi_\sigma(\lambda)$ = output frequencies

$G_\sigma(\lambda)$ = Fourier coefficients

Definition: A *FOT-random systems* is a system that is not FOT-deterministic.

FOT-Deterministic Linear Systems

FOT-Deterministic Systems

$$x_\lambda(t) = e^{j2\pi\lambda t} \;\Rightarrow\; y_\lambda(t) = \sum_{\sigma\in\mathbb{S}} G_\sigma(\lambda)\, e^{j2\pi\varphi_\sigma(\lambda)t}$$

Homogeneity Property of Linear Systems

$$e^{j2\pi\lambda t} X(\lambda)\,\mathrm{d}\lambda \;\Rightarrow\; \sum_{\sigma\in\mathbb{S}} G_\sigma(\lambda)\, e^{j2\pi\varphi_\sigma(\lambda)t} X(\lambda)\,\mathrm{d}\lambda$$

Additivity Property of Linear Systems

$$x(t) = \int_{\mathbb{R}} X(\lambda)\, e^{j2\pi\lambda t}\,\mathrm{d}\lambda \;\Rightarrow\; y(t) = \int_{\mathbb{R}} \sum_{\sigma\in\mathbb{S}} G_\sigma(\lambda)\, e^{j2\pi\varphi_\sigma(\lambda)t} X(\lambda)\,\mathrm{d}\lambda$$

FOT-Deterministic Linear Systems (cont'd)

$$
\begin{aligned}
Y(f) &= \int_{\mathbb{R}} y(t)\, e^{-j2\pi f t}\, \mathrm{d}t \\
&= \int_{\mathbb{R}} \sum_{\sigma \in \mathbb{S}} G_\sigma(\lambda)\, \underbrace{\int_{\mathbb{R}} e^{j2\pi[\varphi_\sigma(\lambda) - f]t}\, \mathrm{d}t}_{\delta(f - \varphi_\sigma(\lambda))}\, X(\lambda)\, \mathrm{d}\lambda \\
&= \int_{\mathbb{R}} \underbrace{\sum_{\sigma \in \mathbb{S}} G_\sigma(\lambda)\, \delta(f - \varphi_\sigma(\lambda))}_{H(f,\lambda)}\, X(\lambda)\, \mathrm{d}\lambda
\end{aligned}
$$

FOT-Deterministic Linear Systems (cont'd)

Transmission Function (Claasen and Mecklenbräuker 1982) (Izzo Napolitano 2002)

$$
\begin{aligned}
H(f,\lambda) &= \sum_{\sigma \in \mathbb{S}} G_\sigma(\lambda)\,\delta(f-\varphi_\sigma(\lambda)) \\
&= \sum_{\sigma \in \mathbb{S}} H_\sigma(f)\,\delta(\lambda-\psi_\sigma(f))
\end{aligned}
$$

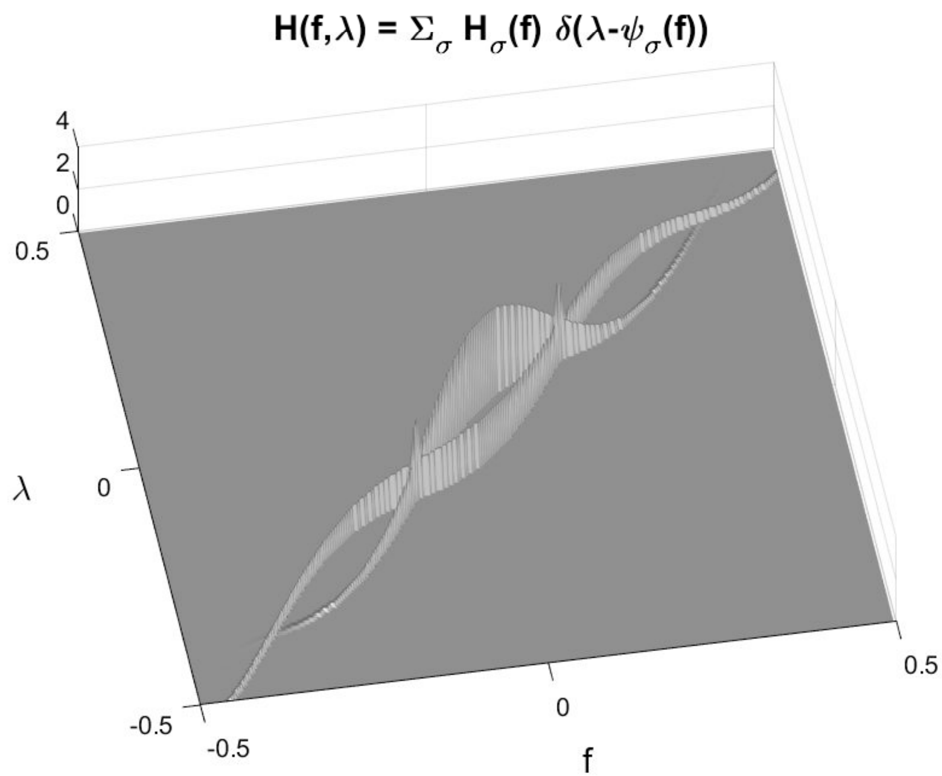
$\psi_\sigma(\cdot)$ = inverse function of $\varphi_\sigma(\cdot)$ (that can always be chosen invertible)
= *frequency mapping function*

$H_\sigma(f) = |\psi'_\sigma(f)|\, G_\sigma(\psi_\sigma(f))$

$G_\sigma(\lambda) = |\varphi'_\sigma(\lambda)|\, H_\sigma(\varphi_\sigma(\lambda))$

$\psi'_\sigma(\cdot)$ and $\varphi'_\sigma(\cdot)$ = first-order derivatives of $\psi_\sigma(\cdot)$ and $\varphi_\sigma(\cdot)$

FOT-Deterministic Linear Systems (cont'd)



FOT-Deterministic Linear Systems (cont'd)

Impulse Response Function (Izzo Napolitano 2002)

$$\begin{aligned} h(t,u) &\triangleq \int_{\mathbb{R}^2} H(f,\lambda)\, e^{j2\pi(ft-\lambda u)}\, \mathrm{d}f\, \mathrm{d}\lambda \\ &= \sum_{\sigma\in\mathbb{S}} \int_{\mathbb{R}} G_\sigma(\lambda)\, e^{j2\pi\varphi_\sigma(\lambda)t}\, e^{-j2\pi\lambda u}\, \mathrm{d}\lambda \\ &= \sum_{\sigma\in\mathbb{S}} \int_{\mathbb{R}} H_\sigma(f)\, e^{-j2\pi\psi_\sigma(f)u}\, e^{j2\pi ft}\, \mathrm{d}f \end{aligned}$$

FOT-Deterministic Linear Systems (cont'd)

Input/Output Relationship in the Frequency Domain

$$\begin{aligned} Y(f) &= \sum_{\sigma \in \mathbb{S}} \int_{\mathbb{R}} G_\sigma(\lambda)\, \delta(f - \varphi_\sigma(\lambda))\, X(\lambda)\, \mathrm{d}\lambda \\ &= \sum_{\sigma \in \mathbb{S}} H_\sigma(f)\, X\left(\psi_\sigma(f)\right) \end{aligned}$$

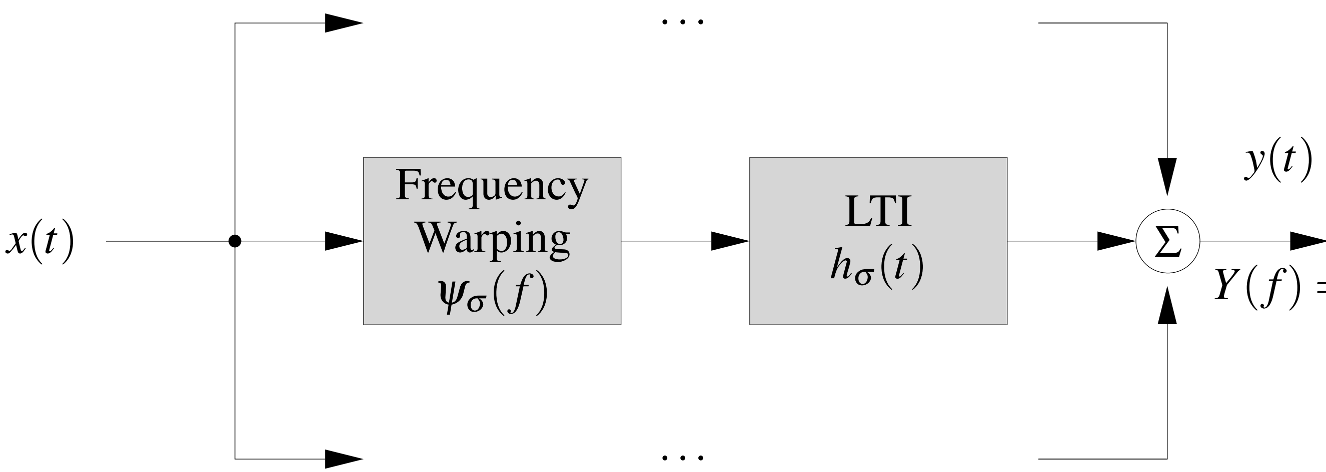
Input/Output Relationship in the Time Domain

$$\begin{aligned} y(t) &= \sum_{\sigma \in \mathbb{S}} \int_{\mathbb{R}} G_\sigma(\lambda)\, X(\lambda)\, e^{j2\pi\varphi_\sigma(\lambda)t}\, \mathrm{d}\lambda \\ &= \sum_{\sigma \in \mathbb{S}} h_\sigma(t) \otimes x_{\psi_\sigma}(t) \end{aligned}$$

$x_{\psi_\sigma}(t)$ **= *Frequency-Warped Version of*** $x(t)$

$$x_{\psi_\sigma}(t) \triangleq \int_{\mathbb{R}} X\left(\psi_\sigma(f)\right) e^{j2\pi f t}\, \mathrm{d}f$$

FOT-Deterministic Linear Systems (cont'd)



$$y(t) = \sum_{\sigma \in \mathbb{S}} h_\sigma(t) \otimes x_{\psi_\sigma}(t)$$

$$Y(f) = \sum_{\sigma \in \mathbb{S}} H_\sigma(f) X(\psi_\sigma(f))$$

$$h_\sigma(t) \stackrel{\mathcal{F}}{\longleftrightarrow} H_\sigma(f)$$

FOT-Deterministic Linear Systems (cont'd)

Examples of FOT-Deterministic LTV Systems:

- Linear Almost-Periodically Time-Variant (LAPTV) Systems

$$\psi_\sigma(f) = f + \sigma \qquad \sigma \in \mathbb{S} \text{ set of frequency shifts}$$

$$y(t) = \sum_{\sigma \in \mathbb{S}} \left[x(t)\, e^{j2\pi\sigma t} \right] \otimes h_\sigma(t)$$

- Linear Time-Invariant Systems

$$\psi_\sigma(f) = f \qquad |\mathbb{S}| = 1$$

$$y(t) = x(t) \otimes h(t)$$

- System Performing a Time-Scale Change

$$h(t,u) = \delta(u - st) \qquad \psi_\sigma(f) = f/s,\ H_\sigma(f) = 1 \qquad |\mathbb{S}| = 1$$

$$y(t) = x(st)$$

- Decimators and interplators are discrete-time deterministic LTV systems.

FOT-Deterministic Linear Systems (cont'd)

Examples of FOT-Random LTV Systems:

- Product Modulation

$$h(t,u) = c(t)\,\delta(u-t)$$

with $c(t)$ almost-cyclostationary (not periodic or almost-periodic) or with finite energy.

$$y(t) = c(t)\,x(t)$$

- Time Warping

$$h(t,u) = \delta(u-\zeta(t))$$

with $\zeta(t)$ not almost-periodic or affine.

$$y(t) = x(\zeta(t))$$

FOT-Deterministic Linear Systems (cont'd)

Remarks:

- The parallel and cascade concatenation of FOT-deterministic LTV systems is still a FOT-deterministic LTV system
- An FOT-deterministic linear system is *causal* if and only if it is LAPTV (Claasen and Mecklenbräuker 1982)
- If $|\mathbb{S}| = 1$ (only one branch), in the *stochastic approach*, FOT-deterministic linear systems transform input wide-sense stationary stochastic processes into output wide-sense stationary processes (Franaszek 1967), (Franaszek and Liu 1967)

Linear System Decomposition into FOT-Deterministic and Residual Components

Stochastic Approach

Decomposition of a Random Process into Deterministic and Residual Components

$$X(t;\omega) = \mathrm{E}\{X(t;\omega)\} + X_r(t;\omega) \qquad t \in \mathbb{R},\ \omega \in \Omega_x$$

deterministic component = $\mathrm{E}\{X(t;\omega)\}$ = expected value of $X(t;\omega)$

residual component = $X_r(t;\omega)$ with $\mathrm{E}\{X(t;\omega)\} = 0$

Decomposition of a Random Linear System into Deterministic and Residual Components

$$h(t,u;\omega) = \mathrm{E}\{h(t,u;\omega)\} + h_r(t,u;\omega) \qquad t,u \in \mathbb{R},\ \omega \in \Omega_h$$

deterministic component = $\mathrm{E}\{h(t,u;\omega)\}$ = expected value of $h(t,u;\omega)$

residual component = $h_r(t,u;\omega)$ with $\mathrm{E}\{h_r(t,u;\omega)\} = 0$

Linear System Decomposition into FOT-Deterministic and Residual Components (cont'd)

Under the assumption that the stochastic process $\{X(t;\omega_x), t \in \mathbb{R}, \omega_x \in \Omega_x\}$ and the stochastic system $\{h(t,u;\omega_h), t,u \in \mathbb{R}, \omega_h \in \Omega_h\}$ are *uncorrelated* ($\mathrm{E}\{X\,h\} = \mathrm{E}\{X\}\,\mathrm{E}\{h\}$)

$$\begin{aligned}
\mathrm{E}\{Y(t;\omega_x,\omega_h)\} &= \mathrm{E}\left\{\int_{\mathbb{R}} h(t,u;\omega_h)\,X(u;\omega_x)\,\mathrm{d}u\right\} \\
&= \mathrm{E}\left\{\int_{\mathbb{R}} \Big[\mathrm{E}\{h(t,u;\omega_h)\} + h_r(t,u;\omega_h)\Big]\,\Big[\mathrm{E}\{X(u;\omega_x)\} + X_r(u;\omega_x)\Big]\,\mathrm{d}u\right\} \\
&= \int_{\mathbb{R}} \mathrm{E}\{h(t,u;\omega_h)\}\,\mathrm{E}\{X(u;\omega_x)\}\,\mathrm{d}u
\end{aligned}$$

Three terms disappear since $\mathrm{E}\{X_r\} = 0$, $\mathrm{E}\{h_r\} = 0$, $\mathrm{E}\{X_r\,h_r\} = 0$.

Linear System Decomposition into FOT-Deterministic and Residual Components (cont'd)

FOT Approach

Decomposition of a Signal into Deterministic and Residual Components

$$x(t) = \mathrm{E}^{\{\alpha\}}\left\{x(t)\right\} + x_r(t)$$

deterministic component = $\mathrm{E}^{\{\alpha\}}\left\{x(t)\right\}$ = almost-periodic component of $x(t)$ = expected value of $x(t)$

residual component = $x_r(t)$ = residual term not containing any finite-strength additive sine-wave component

Decomposition of a Linear System into Deterministic and Residual Components

$$\boxed{h(t,u) = h_{\mathrm{D}}(t,u) + h_r(t,u)}$$

deterministic component = $h_{\mathrm{D}}(t,u)$ = **FOT-deterministic system.**
It cannot be defined in terms the almost-periodic component extraction operator.

residual component = $h_r(t,u) \triangleq h(t,u) - h_{\mathrm{D}}(t,u)$ = **residual term**

Linear System Decomposition into FOT-Deterministic and Residual Components (cont'd)

$$
\begin{aligned}
\int_{\mathbb{R}} h(t,u)\, x(u)\, \mathrm{d}u &= \int_{\mathbb{R}} \Big[h_{\mathrm{D}}(t,u) + h_r(t,u) \Big] \Big[\mathrm{E}_u^{\{\alpha\}}\{x(u)\} + x_r(u) \Big] \mathrm{d}u \\
&= \int_{\mathbb{R}} h_{\mathrm{D}}(t,u)\, \mathrm{E}_u^{\{\alpha\}}\{x(u)\}\, \mathrm{d}u && \text{(a)} \\
&\quad + \int_{\mathbb{R}} h_{\mathrm{D}}(t,u)\, x_r(u)\, \mathrm{d}u && \text{(b)} \\
&\quad + \int_{\mathbb{R}} h_r(t,u)\, \mathrm{E}_u^{\{\alpha\}}\{x(u)\}\, \mathrm{d}u && \text{(c)} \\
&\quad + \int_{\mathbb{R}} h_r(t,u)\, x_r(u)\, \mathrm{d}u && \text{(d)}
\end{aligned}
$$

Linear System Decomposition into FOT-Deterministic and Residual Components (cont'd)

(a) $h_{\mathrm{D}}(t,u)$ transforms an almost-periodic input signal into an almost-periodic output signal
$\Rightarrow$ **term (a) is almost-periodic**

(b) $x_r(u)$ does not contain any finite-strength additive sinewave component
$\Rightarrow$ passing through $h_{\mathrm{D}}(t,u)$ does not produce any output finite-strength sine wave
$\Rightarrow$ **term (b) does not contain any almost-periodic component**

(c) $h_r(t,u)$ is the residual term with respect to $h_{\mathrm{D}}(t,u)$
$\Rightarrow$ if its input is almost-periodic its output does not contain any finite-strength sine wave
$\Rightarrow$ **term (c) does not contain any almost-periodic component**

(d) If $h_r(t,u)$ is *functionally independent* of $x_r(u)$, that is, there is no operator $\mathscr{H}$ such that $h_r(t,u) = \mathscr{H}[x_r(\cdot);t,u]$, thus from the residual term $x_r(u)$ no hidden higher-order almost-periodic component can be regenerated by $h_r(t,u)$
$\Rightarrow$ **term (d) does not contain any almost-periodic component**

Linear System Decomposition into FOT-Deterministic and Residual Components (cont'd)

$$
\begin{aligned}
\mathrm{E}_t^{\{\alpha\}}\{y(t)\} &= \mathrm{E}_t^{\{\alpha\}}\left\{\int_{\mathbb{R}} h(t,u)\,x(u)\,\mathrm{d}u\right\} \\
&= \mathrm{E}_t^{\{\alpha\}}\left\{\int_{\mathbb{R}} \Big[h_{\mathrm{D}}(t,u)+h_r(t,u)\Big]\,\Big[\mathrm{E}_u^{\{\alpha\}}\{x(u)\}+x_r(u)\Big]\,\mathrm{d}u\right\} \\
&= \int_{\mathbb{R}} h_{\mathrm{D}}(t,u)\,\mathrm{E}_u^{\{\alpha\}}\{x(u)\}\,\mathrm{d}u
\end{aligned}
$$

$h_{\mathrm{D}}(t,u)$ is the kernel of the linear operator that transforms the almost-periodic component of the input signal into the almost-periodic component of the output signal.

Linear System Decomposition into FOT-Deterministic and Residual Components (cont'd)

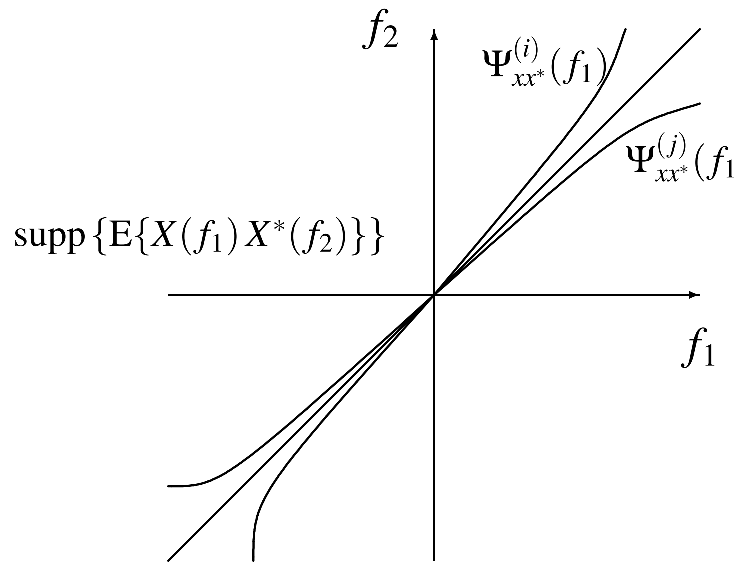
	FOT deterministic	*FOT random*	*residual term*
signals	almost-periodic (AP) functions	not FOT deterministic	not containing any finite-strength additive sine-wave component
systems	transform AP functions into AP functions	not FOT deterministic	transform AP functions into signals not containing finite-strength additive sine-wave components

FOT-Deterministic LTV Systems and Spectrally Correlated Processes

Spectrally Correlated (SC) Processes

Loève bifrequency spectrum

$$\mathrm{E}\{X(f_1)\,X^*(f_2)\} = \sum_k S_{xx^*}^{(k)}(f_1)\,\delta(f_2 - \Psi_{xx^*}^{(k)}(f_1))$$



Spectral masses are concentrated on a countable set of support curves

FOT-Deterministic LTV Systems and Spectrally Correlated Processes (cont'd)

- Spectrally correlated processes are closed under FOT-deterministic LTV transformations.
- The Wiener filter with desired and input signal that are singularly and jointly spectrally correlated is a FOT-deterministic LTV system.

Conclusion

- In the FOT approach signals can be decomposed into the sum of an almost-periodic component and a residual term non containing any finite-strength additive sine-wave component.
- In the FOT approach linear systems can be decomposed into the parallel of a FOT deterministic component and a residual term.
- The FOT deterministic component transforms input almost-periodic functions into output almost-periodic functions.
- The residual term transforms input sine waves into output signals not containing any finite-strength additive sine-wave component.

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