THE 14TH WORKSHOP ON NONSTATIONARY SYSTEMS AND THEIR APPLICATIONS

# On explicit interpretation of the FOT formal expressions for continuous-time signals and its estimation using sampled finite-time observations

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February 8-10, 2021

Gródek nad Dunajcem [online]

# Outline

- Acknowledgement & Previous works
- Motivation
- Interpretation for ‘Dirac-delta’ notation
- Finite observation -> FOT density
- Estimator based on linear approximation
- Examples
- Conclusion

# Acknowledgement & Previous works

## Great Thanks to Prof. Antonio Napolitano

**University of Napoli "Parthenope", Napoli, Italy**

Previous works that was the main basis for these results

A. Napolitano, **Cyclostationary Processes and Time Series: Theory, Applications, and Generalizations**, Academic Press, 2019.

W. A. Gardner and W. A. Brown, "**Fraction-of-time probability for time-series that exhibit cyclostationarity**," Signal Processing, vol. 23, pp. 273-292, June 1991.

J. Leśkow and A. Napolitano, "**Foundations of the functional approach for signal analysis**," Signal Processing, vol. 86, no. 12, pp. 3796-3825, December 2006.

D. Dehay, J. Leśkow, and A. Napolitano, "**Time average estimation in the fraction-of-time probability framework**," Signal Processing, vol. 153, pp. 275-290, December 2018, doi: 10.1016/j.sigpro.2018.07.005.

# Motivation

Why can we be interested in applying FOT paradigm for finite-length observations?

1. The observation of length $T$ can be the only part of the signal which is ever available for processing.

2. The observation of length $T$ can be used as the part of some much longer (even infinite) observation for approaching infinite-limit averaging.

3. If the processed signal is $T$-periodic, it looks reasonable enough to observe this signal only over the finite interval of length $T$.

4. [**Generalization of 3.**] Some transformations of the signal may well preserve some of its FOT characteristics (or change them in a certain way) so the finite-time observation appears to be a suitable starting point for the signal characterization which may be applied for a possibly infinite observation.

# Finite observation FOT

FOT distribution measured where $\tau \in [t - T/2, t + T/2]$:

$$F_{T,t}(\xi) = \frac{1}{T}\mu\left(\left\{\tau \in [t - T/2, t + T/2] : x(\tau) \le \xi\right\}\right) = \frac{1}{T}\int_{t-T/2}^{t+T/2} \mathbf{1}_{\{x(t)\le\xi\}}\, d\tau$$

FOT distribution measured over finite interval $[0, T]$:

$$F_{x\|[0,T]}(\xi) = \frac{1}{T}\int_0^T u\left(\xi - x(t)\right) dt$$

FOT density measured over finite interval $[0, T]$:

$$f_{x\|[0,T]}(\xi) = \frac{d}{d\xi}F_{x\|[0,T]}(\xi) = \frac{1}{T}\int_0^T \delta\left(\xi - x(t)\right) dt$$

The convention is used: $\delta\left(\xi - x(t)\right) = \frac{d}{d\xi}u\left(\xi - x(t)\right)$

$u(\bullet)$ denotes Heaviside function, $\delta(\bullet)$ denotes Dirac delta-function.

Is it possible to find an explicit interpretation for this FOT density?

For the sake of simplicity, let us assume the finite observation is valid for the rest of the talk: $f_x(\xi) \triangleq f_{x\|[0,T]}(\xi)$ $\quad F_x(\xi) \triangleq F_{x\|[0,T]}(\xi)$

# Conceptual derivation

Let us start from the one-dimensional identity involving Dirac delta

$$\delta\left(g(t)\right) = \sum_{t_m \in R_g} \frac{1}{\left|g'(t_m)\right|}\delta(t - t_m)$$

$R_g$ is the set of $\{t_m\}$ where $g(t) = 0$

Two potential problems may easily appear about this formula:

1. What should we do if $g'(t)=0$ at some isolated point?
2. What should we do if function $g(t)$ is so as for each $t \in [t_1, t_2]$, $g'(\mathrm{t})=0$, e.g., $g(t)$ remains flat in the solid interval?

However, $\delta\left(\xi - x(t)\right)$ is actually a two-dimensional function of $\xi$ and $t$ which is obtained via:

$$\delta\left(\xi - x(t)\right) = \frac{d}{d\xi} u\left(\xi - x(t)\right)$$

The accurate consideration of the ‘solid’ intervals leads to the proposed inference.

# Root sets

$\delta\left(\xi - x(t)\right)$ can be considered as a function of two variables ($\xi$, $t$).

'Something' non zero will exist only at such points where: $\xi - x(t) = 0$

They belong to the set of roots: $R_x(\xi) = \left\{t : x(t) = \xi\right\}$

Two types of roots are identified: ***isolated*** and ***non isolated***.

***Isolated*** root $t_0 \in R_x(\xi)$ is defined as follows:

$$\exists \varepsilon > 0 \quad \forall t \in (t_0 - \varepsilon; t_0 + \varepsilon): \quad t_0 \in R_x(\xi) \wedge t \notin R_x(\xi)$$

Otherwise root $t_0$ will be called *non isolated*.

Assumption 1:

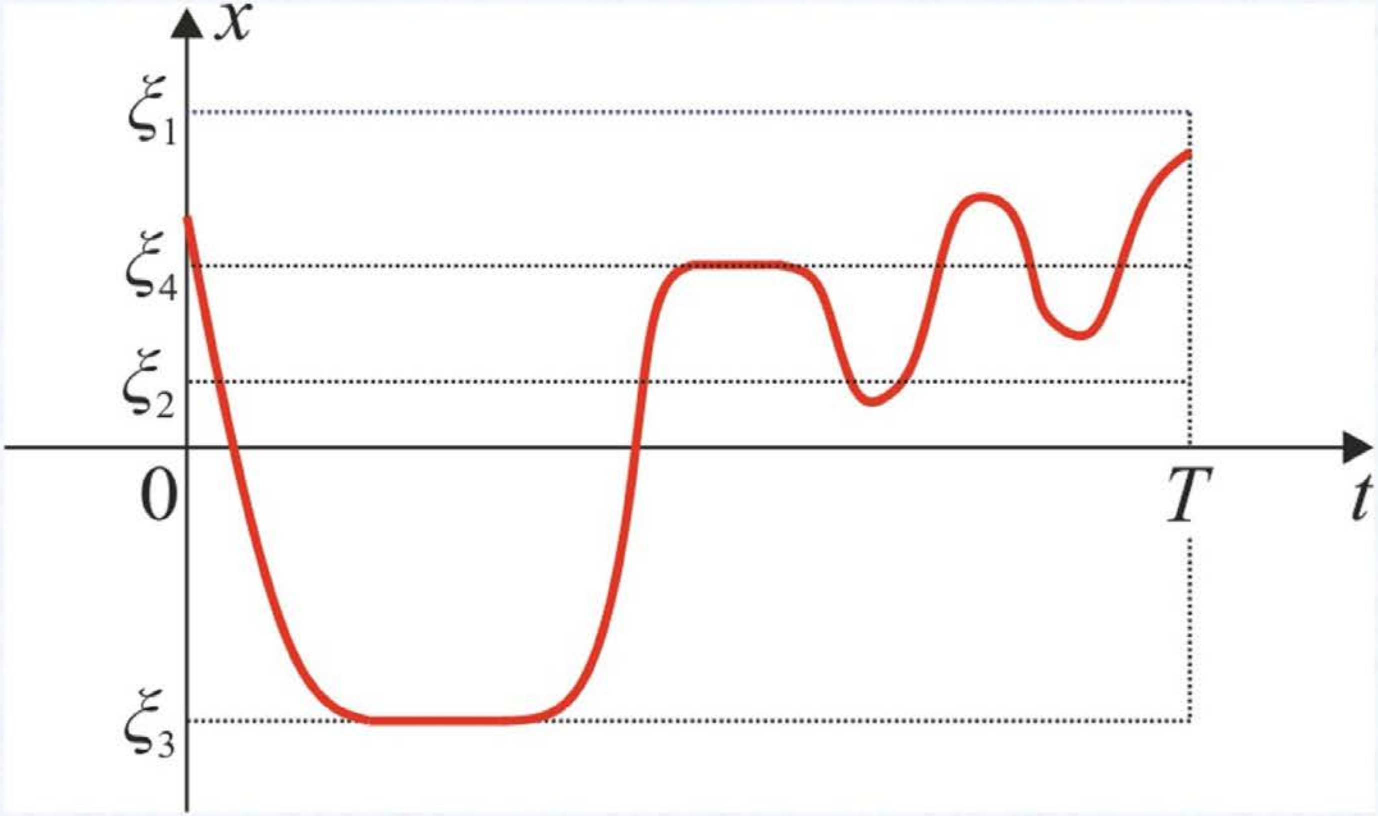
For each $\xi$, all associated *isolated* roots form a countable set, which is denoted by $\{t_m(\xi)\}$.

The set of *non-isolated* roots is $Ru_x(\xi) = R_x(\xi) \setminus \{t_m(\xi)\}$.

Assumption 2:

All values $\xi$ for each the set of *non-isolated* roots is non empty $Ru_x(\xi) \neq \varnothing$ form countable set $\{x_k\}$.

# An example (root sets)




$R_x(\xi_1) = \varnothing$

$R_x(\xi_2)$ *is made of isolated roots only*

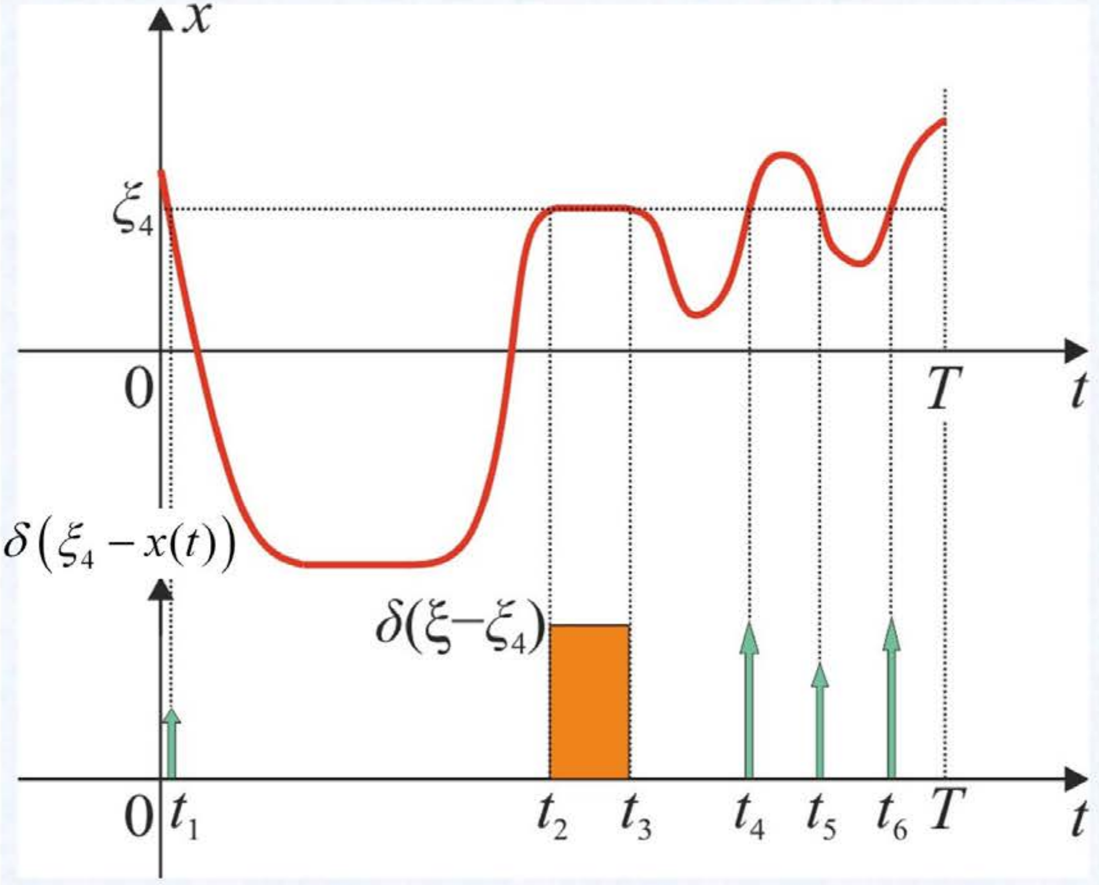
$R_x(\xi_3)$ *is made of non-isolated roots only*

$R_x(\xi_4)$ *is the set where roots of both types are present*

# Time function representation

For fixed value $\xi$ we can consider $\delta(\xi - x(t))$ as **a function of time *t***:

$$\delta\left(\xi - x(t)\right) = \sum_m \frac{1}{\left|x'\left[t_m(\xi)\right]\right|}\delta\left(t - t_m(\xi)\right) + \sum_k \mathbf{1}[t \in Ru_x(x_k)]\delta(\xi - x_k)$$




For chosen $\xi_4$:

*non-isolated roots:*

$$Ru\left(\xi_4\right) = [t_2, t_3]$$

*all values in the interval*

*isolated roots:*

$$\{t_m(\xi_4)\} = \{t_1, t_4, t_5, t_6\}$$

*All roots:*

$$R(\xi_4) = Ru(\xi_4) \cup \{t_m(\xi_4)\}$$

# Finite-time FOT density

The integration $\delta\left(\xi - x(t)\right)$ over finite interval $[0, T]$ will result in

$$f_x(\xi) = \frac{1}{T}\int_0^T \delta\left[\xi - x(t)\right]dt = \frac{1}{T}\sum_m \frac{1}{\left|x'\left[t_m(\xi)\right]\right|} + \frac{1}{T}\sum_k \mu\left(t \in [0,T] : x(t) = \xi\right)\delta(\xi - x_k)$$

$\mu(\bullet)$ is Lebesgue measure

The FOT density consists of (at least) two parts.

The intervals where the signal remains constant (or ‘flat’) contribute to the part of FOT density which is represented by Dirac delta-functions, or concentrated FOT masses.

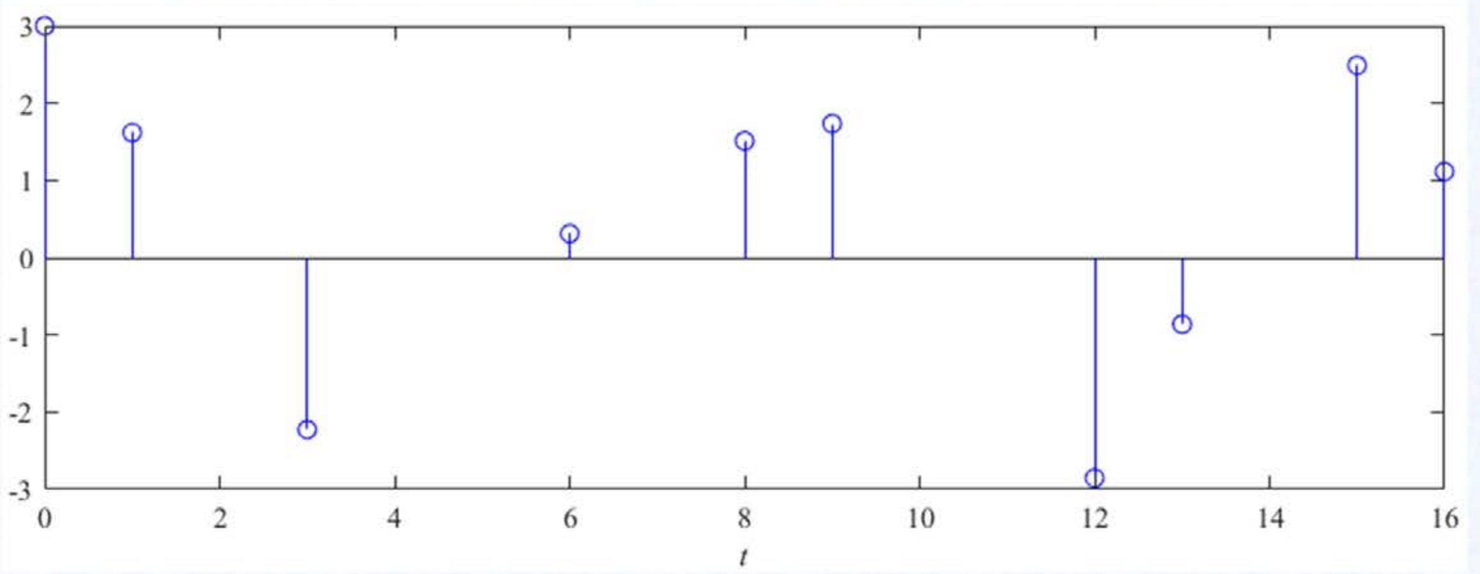
The intervals where signal is varying in time contribute to the continuous part of the FOT density.

# An estimator for CT based on time series

Suppose we observe a finite time series representing the continuous signal over the finite interval $[0, T] = [t_0, t_{N-1}]$:

$$x_n = x(t_n) \qquad \{(t_n, x_n)\}, n \in \mathbb{Z} : 0 \le n \le N-1$$

Time instants $t_n$ are not necessary to be so they produce uniform sampling.



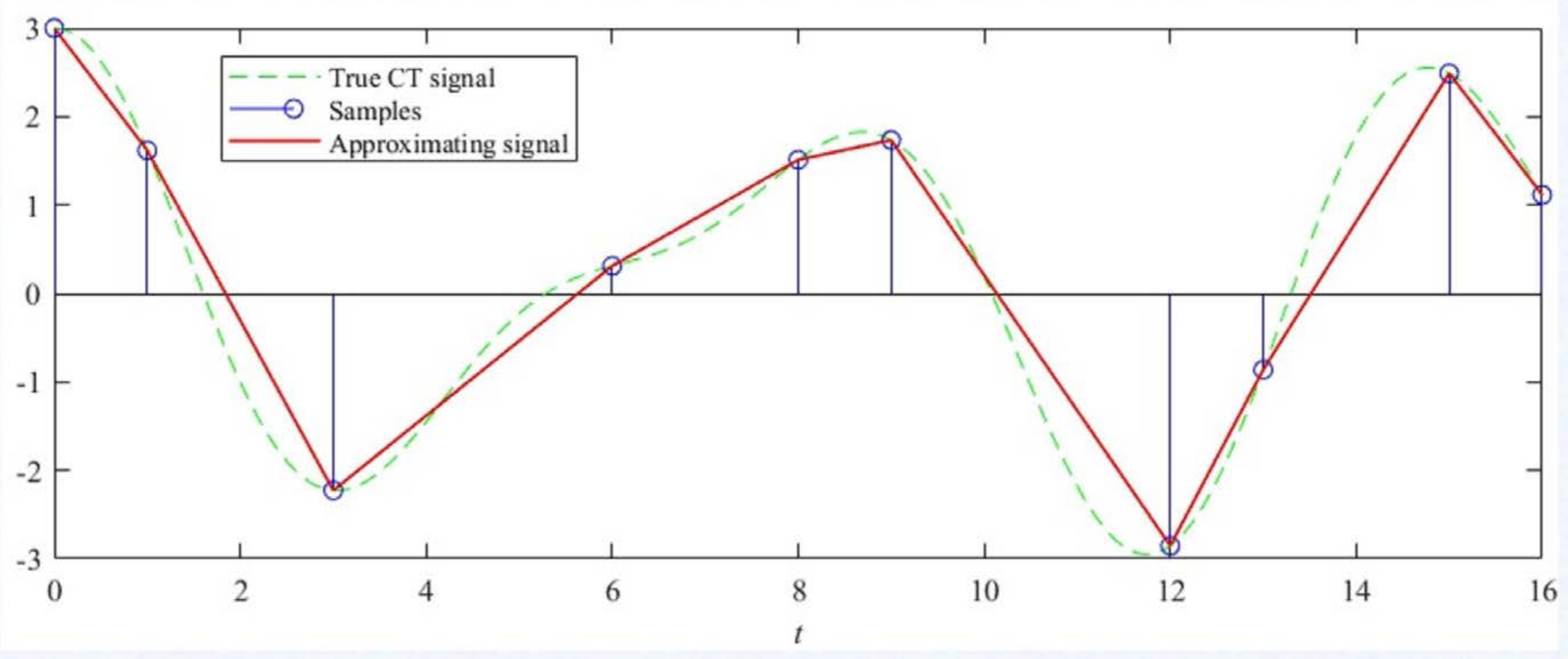

$$\hat{f}_x(\xi) = \sum_{n=1}^{N-1} w_n \hat{f}_x(\xi; n) \qquad w_n = \frac{\mu([t_{n-1}, t_n])}{\mu([t_0, t_{N-1}])} = \frac{t_n - t_{n-1}}{T} \qquad \sum_{n=1}^{N} w_n = 1$$

$f_x(\xi; n)$ - FOT density estimated for $n$th segment

# Linear approximation

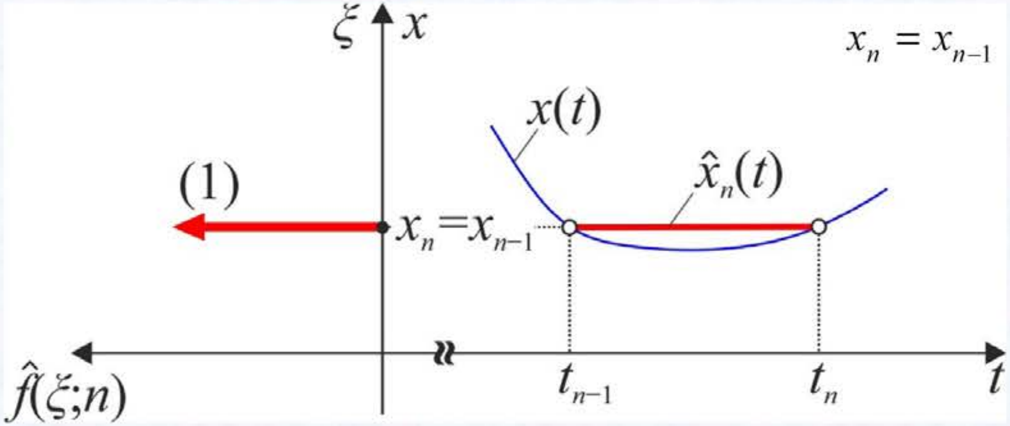
One of the simplest approximation is the piece-wise linear approximation, where the reconstructed signal within the $n$th interval can be estimated as the line:

$$\hat{x}(t;n) = x_{n-1} + \frac{x_n - x_{n-1}}{t_n - t_{n-1}}(t - t_{n-1}), \quad t_{n-1} \le t \le t_n$$



# FOT estimation for a linear segment

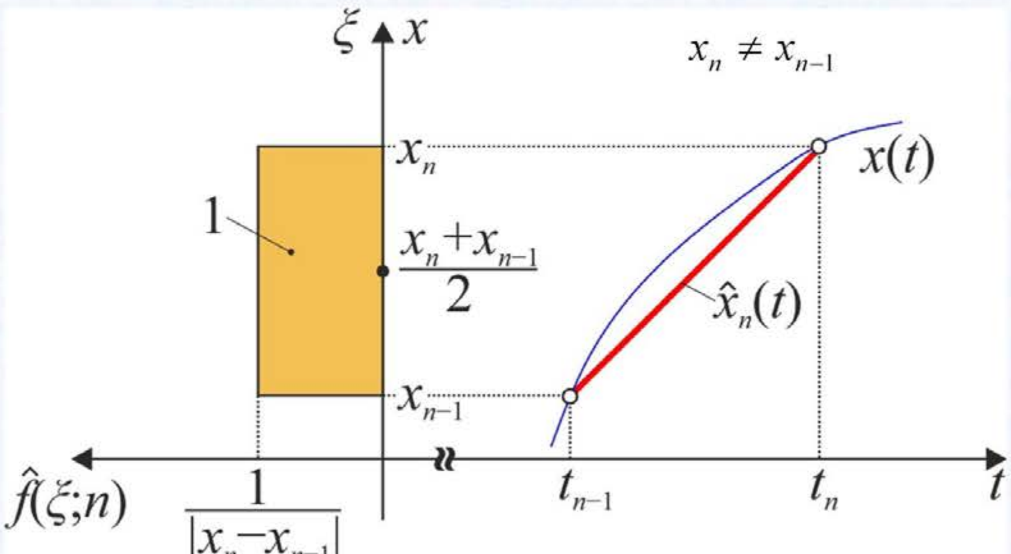
Case 1: A constant approximation




$$\hat{f}_x(\xi;n) = \delta(\xi - x_n)$$

*Same formula is valid for positive and negative slopes*

Case 2: A non-zero slope linear approximation




$$\hat{f}(\xi;n) = \frac{1}{|x_n - x_{n-1}|}\text{rect}\left(\frac{\xi - \frac{x_n + x_{n-1}}{2}}{|x_n - x_{n-1}|}\right)$$

$$\text{rect}(v) = \begin{cases} 1, & \text{if } |v| < 0.5, \\ 0.5, & \text{if } |v| = 0.5, \\ 0, & \text{if } |v| > 0.5 \end{cases}$$

# FOT density of the approximating signal

Generally, for any segment of the approximating signal, the equation will hold:

$$\hat{f}_x(\xi;n)=\begin{cases}\dfrac{1}{|x_n-x_{n-1}|}\operatorname{rect}\left(\dfrac{\xi-\dfrac{x_n+x_{n-1}}{2}}{|x_n-x_{n-1}|}\right), & \text{if } x_n\neq x_{n-1},\\ \delta(\xi-x_n), \quad \text{if } x_n=x_{n-1}, \end{cases}$$

The weighted sum for all intervals yields the total finite-time FOT density of the approximating signal:

$$\hat{f}_x(\xi)=\sum_{n=1}^{N-1} w_n \hat{f}_x(\xi;n)$$

If uniform sampling with period $T_s$ takes place $t_n=nT_s$ :

$$\hat{f}_x(\xi)=\frac{1}{N-1}\sum_{n=1}^{N-1}\hat{f}_x(\xi;n)$$

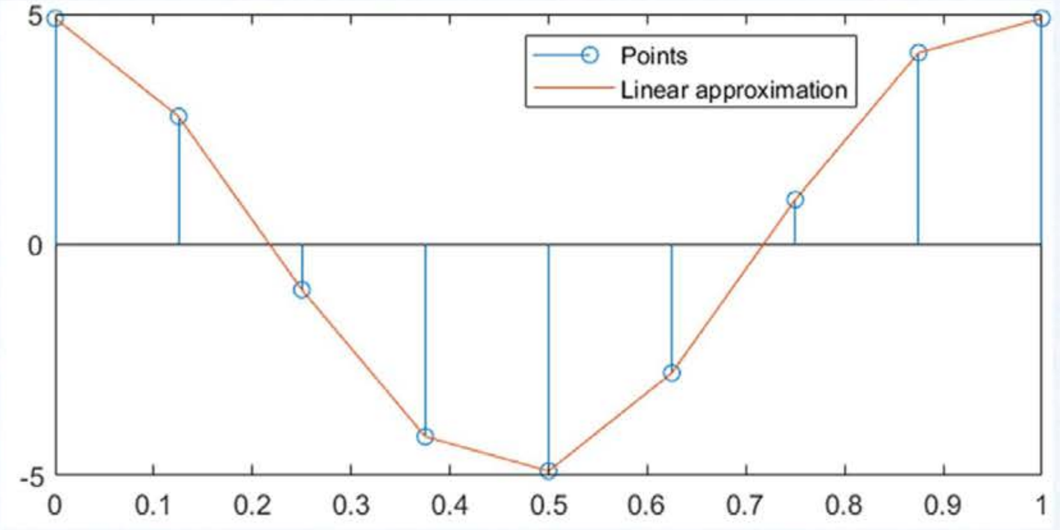
# Example 1-1: One period

$$x[n] = A\cos\left(\frac{\pi}{4}n + \frac{\pi}{16}\right)$$

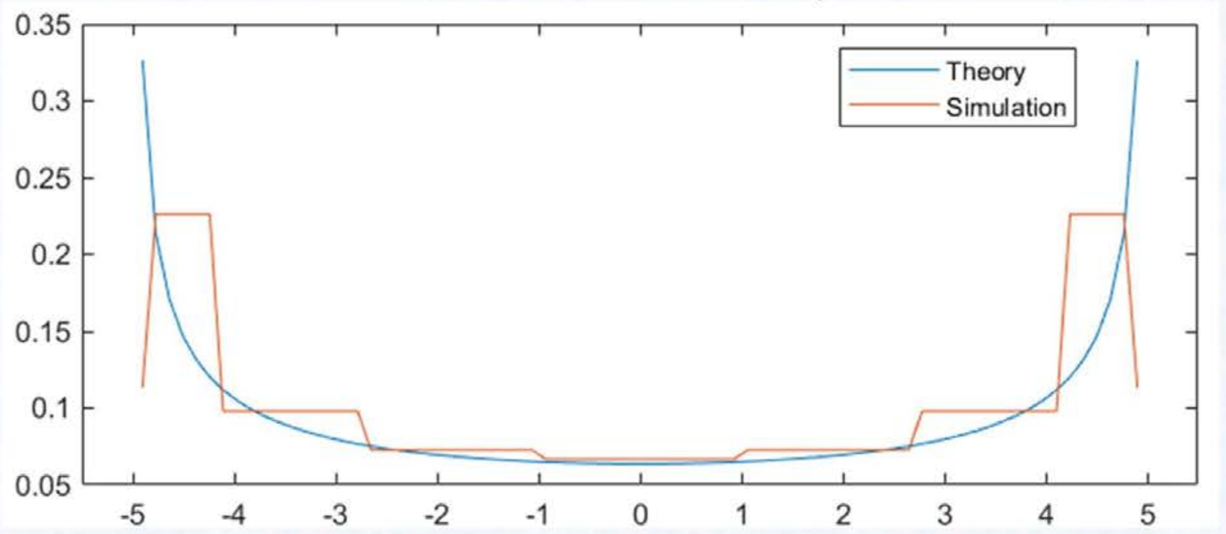
$$0 \le n \le 8, \quad T_s = 1/8$$

$$f_x(\xi) = \frac{1}{\pi A}\frac{1}{\sqrt{1-\left(\xi/A_0\right)^2}}\,\mathrm{rect}\left(\xi/2A\right)$$

Sample and its approximation




Estimated FOT density



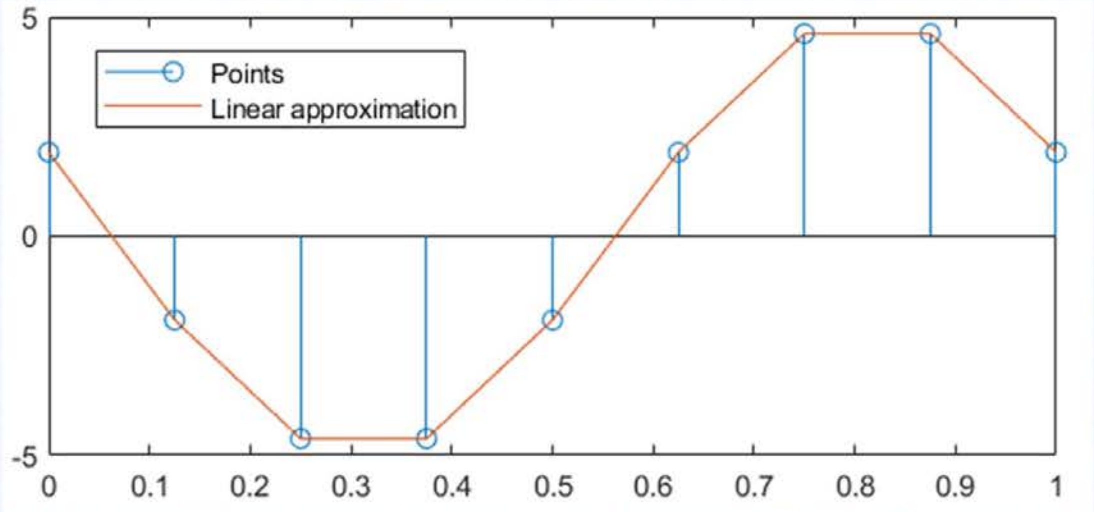
# Example 1-2: One period

$$x[n] = A\cos\left(\frac{\pi}{4}n + \frac{3\pi}{8}\right)$$

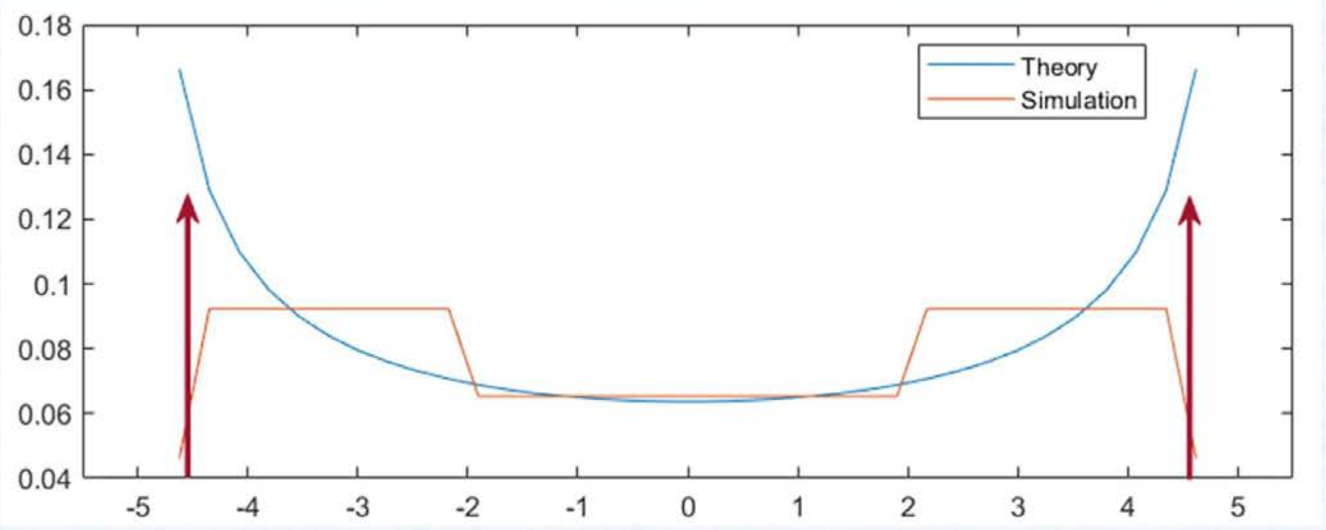
$$0 \le n \le 8, \quad T_s = 1/8$$

$$f_x(\xi) = \frac{1}{\pi A}\frac{1}{\sqrt{1-\left(\xi/A_0\right)^2}}\mathrm{rect}\left(\xi/2A\right)$$

Sample and its approximation




Estimated FOT density



# Example 2-1: Long observation - Periodic

$x[n] = A\cos(\beta n + \varphi)$

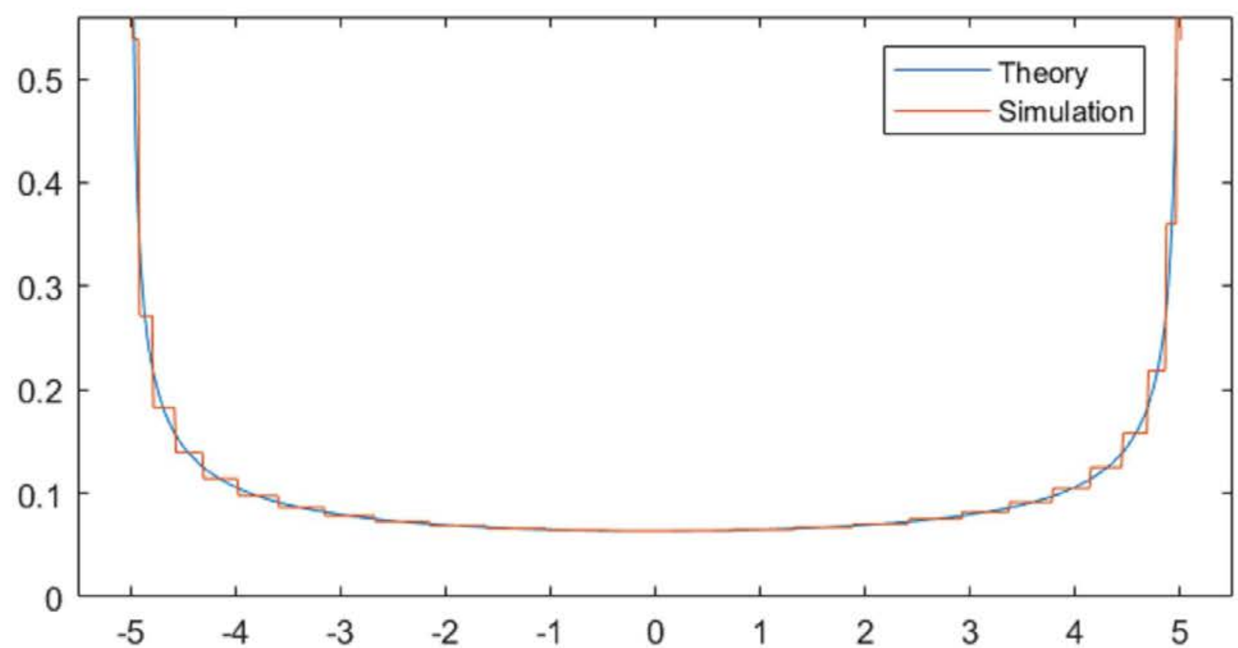
$\beta = \frac{\pi}{53} \quad A = 5 \quad \varphi = 0$

$N = 512\,000$

Theoretical FOT density

$$f_x(\xi) = \frac{1}{\pi A}\frac{1}{\sqrt{1-(\xi/A)^2}}\operatorname{rect}(\xi/2A)$$

Estimated FOT density



# Example 2-2: Long observation - AP

AP time series

$$x[n] = A\cos\left(\beta n + \varphi\right)$$

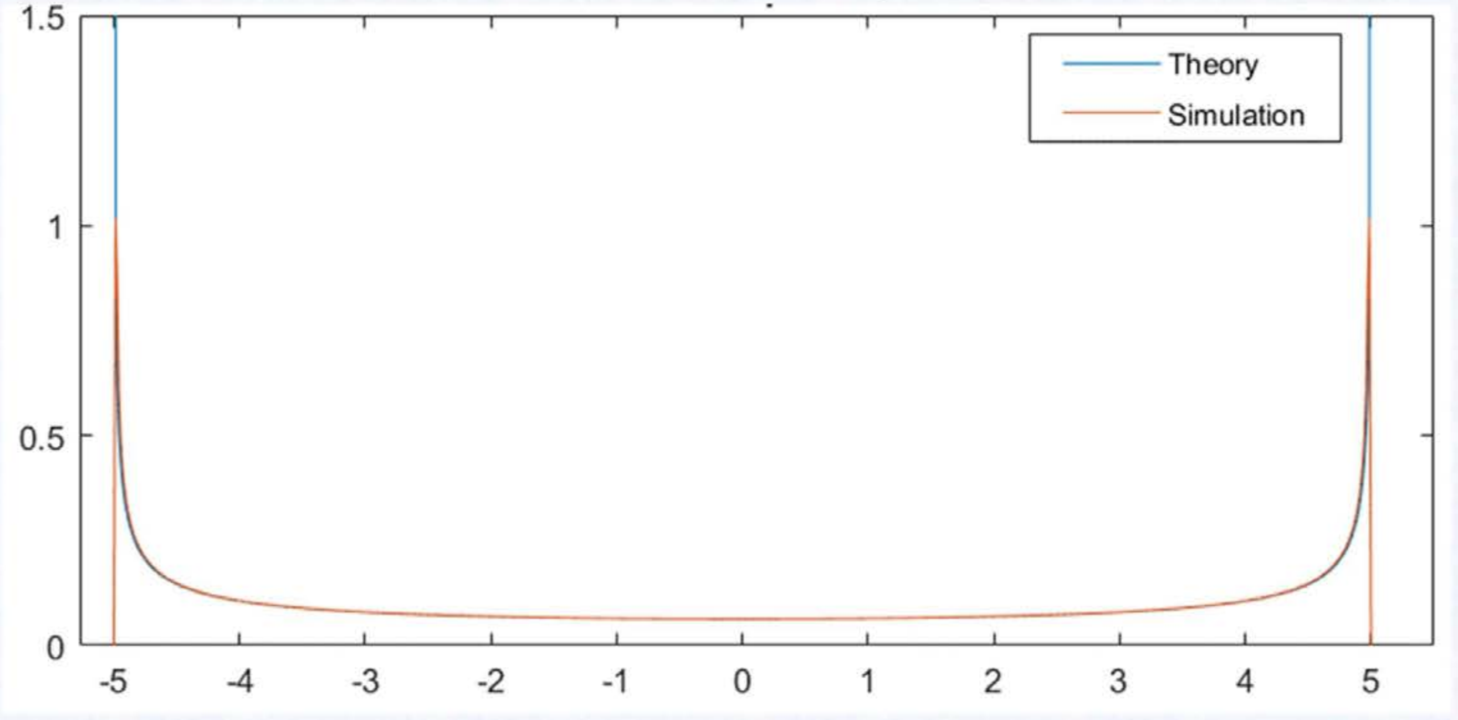
$$\beta = \frac{\pi\sqrt{2}}{53} \quad A = 5 \quad \varphi = 0$$

$$N = 512\,000$$

Theoretical FOT density

$$f_x(\xi) = \frac{1}{\pi A}\frac{1}{\sqrt{1-\left(\xi/A\right)^2}}\mathrm{rect}\left(\xi/2A\right)$$

Estimated FOT density



# Example 3: Gaussian process

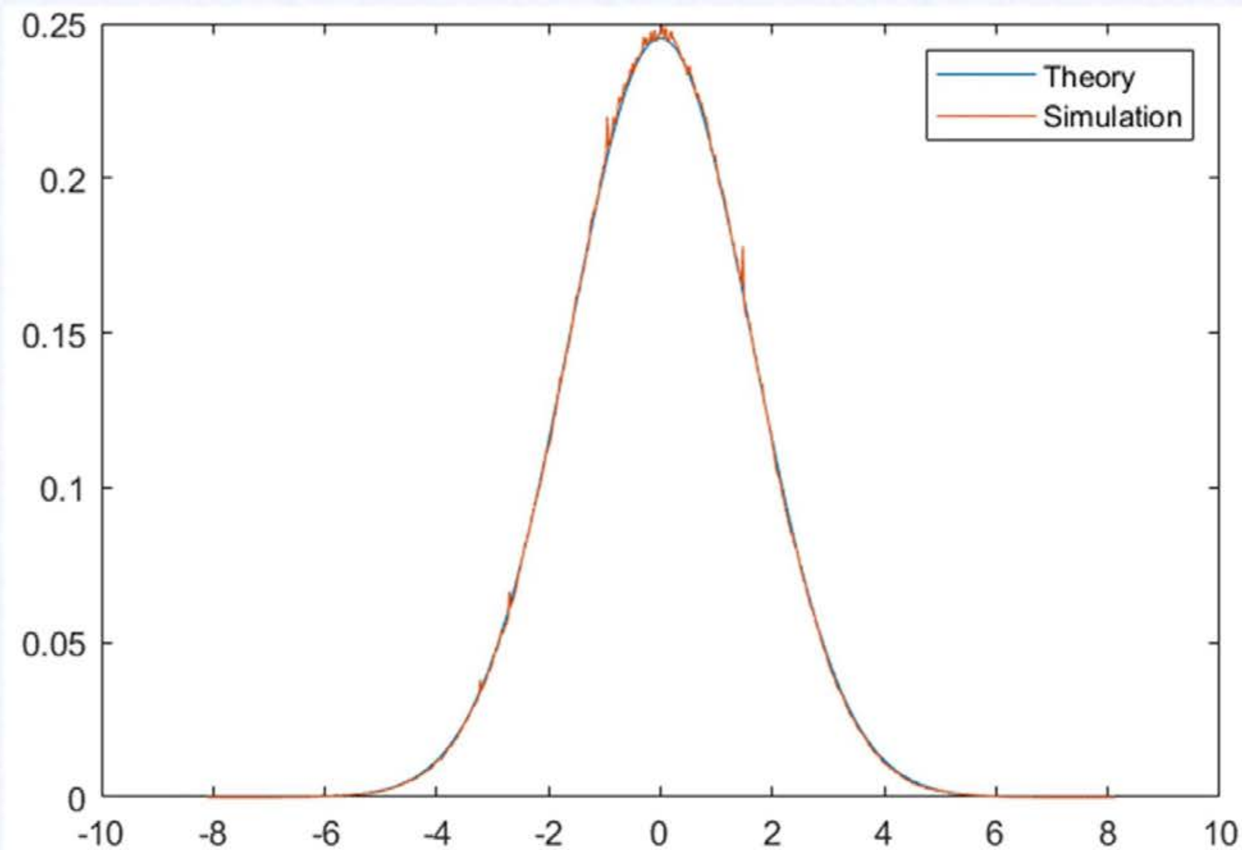
Gaussian process

1st order AR model

$$H(z) = \frac{1}{1-0.8z^{-1}} \qquad N = 65\ 536$$

Theoretical FOT density

$$f_g(\xi) = \frac{1}{\sqrt{2\pi}\sigma_g} e^{-\xi^2/2\sigma_g^2}$$



# Example 4

* The example is proposed and developed by A. Napolitano

Process: $X(t) = G(t)c(t) = G(t)A_0 \cos(2\pi f_0 t + \phi_0)$

$G(t)$ is a zero-mean 1st-order strict-sense stationary Gaussian stochastic process

$A_0, f_0$, and $\phi_0$ are deterministic parameters.

Since $G(t)$ and $c(t)$ are FOT-independent, FOT density of $X(t)$ :

$$f_x(\xi) = \int_{-\infty}^{+\infty} f_c(s) f_g(\xi/s) \frac{1}{|s|} ds \qquad \textit{(Mellin convolution)}$$

$$f_g(\xi) = \frac{1}{\sqrt{2\pi}\sigma_g} e^{-\xi^2/2\sigma_g^2} \qquad f_c(\xi) = \frac{1}{\pi A_0} \frac{1}{\sqrt{1-(\xi/A_0)^2}} \operatorname{rect}(\xi/2A_0)$$

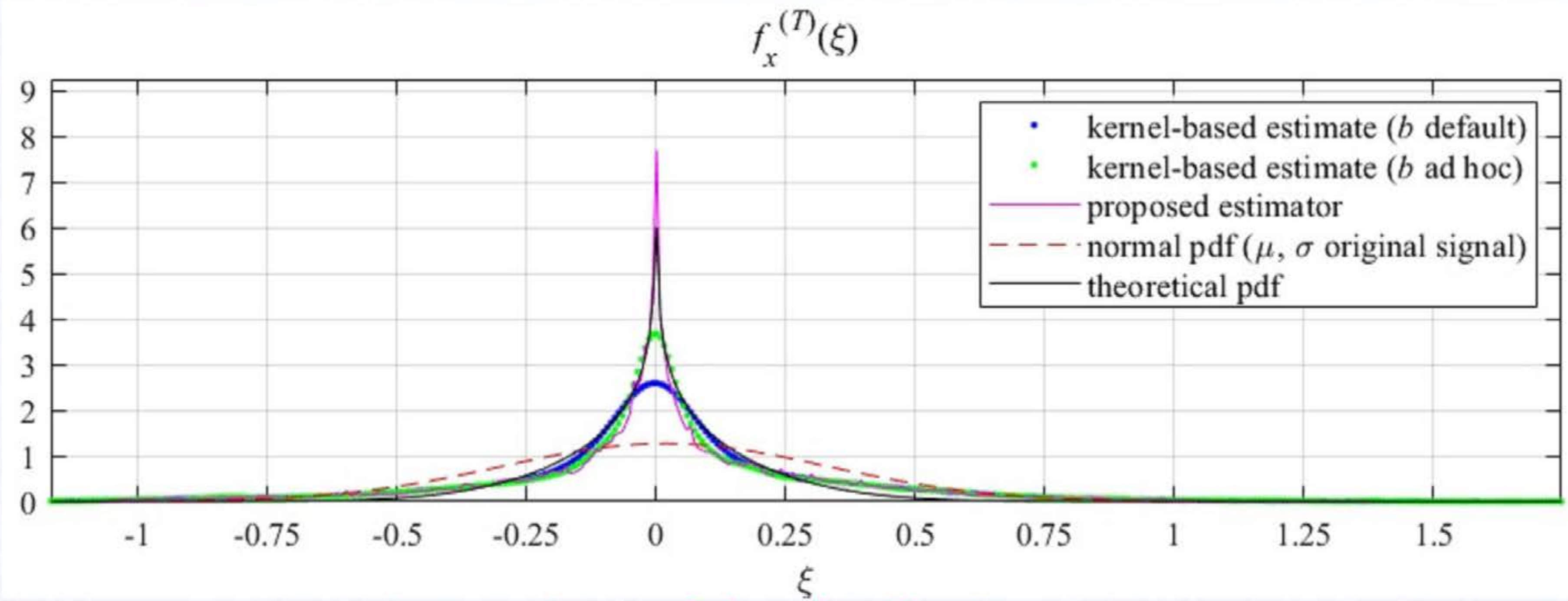
Observation: $x(t) = g(t)A_0 \cos(2\pi f_0 t + \phi_0) \quad t \in [0, T]$

The uniformly sampled observation: $x_n \triangleq x(nT_s) \qquad N=4096$

The comparison between theoretical FOT-density and:

- proposed estimator
- classical kernel-based PDF estimator with Gaussian kernel

# Example 4: density comparison




Comparison of the distributions

| Estimator | Metrics | |
|---|---|---|
| | Kolmogorov-Smirnov test statistics | $L^1$ distance |
| proposed | 1.69 | 0.30 |
| kernel ($b$ ad hoc) | 2.33 | 0.28 |
| kernel ($b$ default) | 3.41 | 0.28 |

Default choice of b due to Silverman's rule

$$b_{default} = (4/3N)^{1/5} s_X$$

$$\widehat{f}_x^{(\text{kernel})}(\xi) \triangleq \sum_{n=0}^{N-1} W((\xi - x_n)/b_N)/(Nb_N)$$

# Conclusion

- The explicit interpretation of $\delta\left(\xi - x(t)\right)$ for continuous time $x(t)$ as a function of time $t$ is considered. The function appears to consist of (*at least*) two parts: one-dimensional Dirac deltas in time and delta in $\xi$ domain with support in $t$ Domain of non-zero Lebesgue measure.
- The given interpretation paved the way to the direct formula of FOT density $f_X(\xi)$ evaluation (or estimation) as a function consisting of (*at least*) two parts: continuous density and mass function (Dirac-deltas)
- A simple estimator for FOT density of CT signal based on its linear approximation where a given time series is available is proposed.
- For double-sideband (DSB) amplitude-modulated (AM) suppressed-carrier stochastic process with Gaussian modulating function, the estimator showed better performance measured by Kolmogorov-Smirnov distance and practically the same performance of the other estimators in terms of $L^1$ distance.