FOT frequency extraction from the distribution of a signal

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February 8–10, 2021,

The Fourteenth Workshop on Nonstationary Systems
and Their Applications
Gródek nad Dunajcem, Poland

Introduction

Example 1: Periodic function

Let $z(t)$ be a bounded periodic function with period τ_1.

Then $\mathbb{I}_{\{z(t)\leq\xi_0\}}$ is periodic with the same period τ_1 for any $\xi_0 \in \mathbb{R}$.

Let $\gamma_1 \triangleq \tau_1^{-1}$ (fundamental frequency) and $\Lambda_1 \triangleq \gamma_1\mathbb{Z}$.

$$a_z^\lambda \triangleq \left\langle z(t)\, e^{-j2\pi\lambda t}\right\rangle_t \triangleq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} z(t)\, e^{-j2\pi\lambda t}\, dt, \quad \lambda \in \mathbb{R}.$$

Then $a_z^\lambda = 0$ if $\lambda \notin \Lambda_1$ and

$$a_z^\lambda = \frac{1}{\tau_1}\int_0^{\tau_1} z(t)\, e^{-j2\pi\lambda t}\, dt \quad \text{if } \lambda \in \Lambda_1.$$

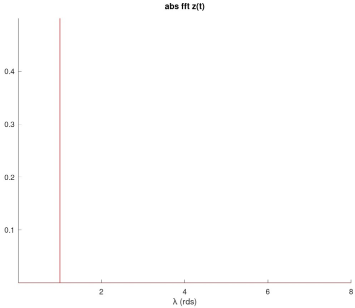
The set of frequencies of $z(t)$ is defined by $\Gamma_z \triangleq \{\lambda \in \mathbb{R} : a_z^\lambda \neq 0\} \subset \Lambda_1$.

Then $\Gamma_{\mathbb{I}_{\{z(t)\leq\xi_0\}}} \subset \Lambda_1$.

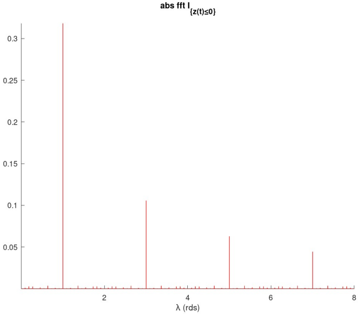
Question : Link between $\Gamma_{\mathbb{I}_{\{z(t)\leq\xi_0\}}}$ and Γ_z ?

$$z(t) = \cos(2\pi t)$$

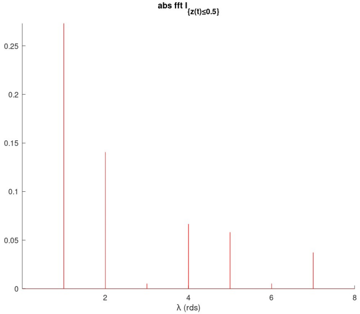
abs fft $\cos(2\pi t)$



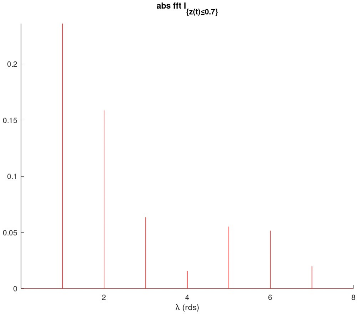
abs fft $\mathbb{I}_{\{\cos(2\pi t)\leq 0\}}$



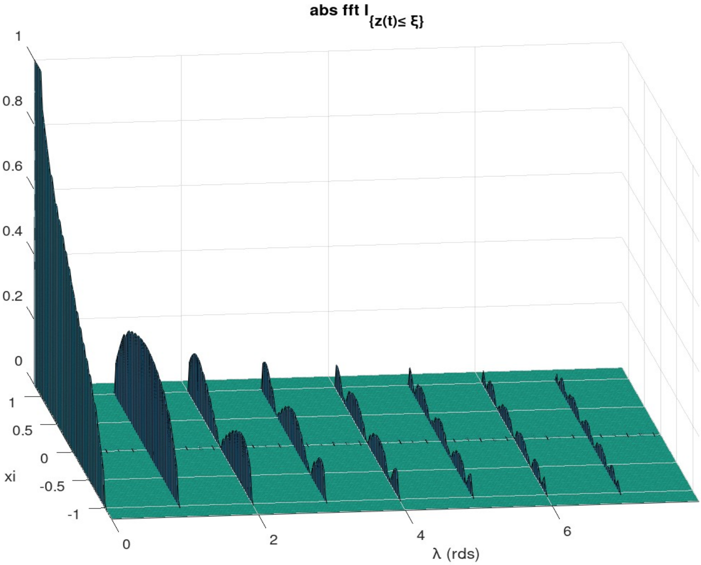
abs fft $\mathbb{I}_{\{\cos(2\pi t)\leq 0.5\}}$



abs fft $\mathbb{I}_{\{\cos(2\pi t)\leq 0.7\}}$



abs fft $\mathbb{I}_{\{\cos(2\pi t)\leq\xi\}}$

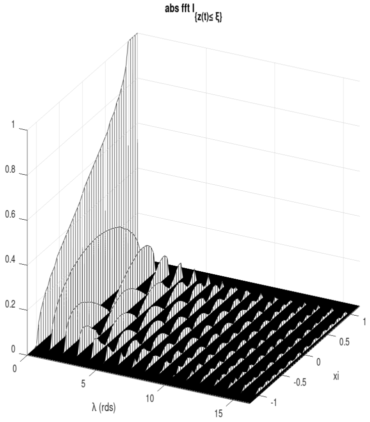


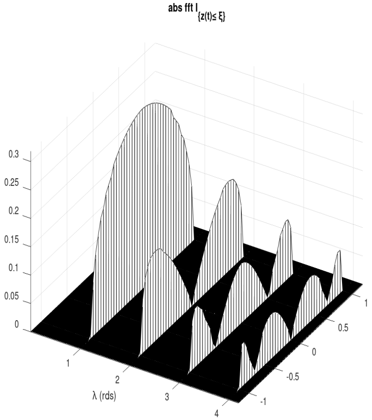
Apparition of **new harmonics** for the indicator function :

$$\gamma_k = k, \qquad k \in \mathbb{Z} \qquad !$$

abs fft $\mathbb{I}_{\{\cos(2\pi t)\leq\xi\}}$

$\lambda \neq 0$





Example 2: Poly-periodic function

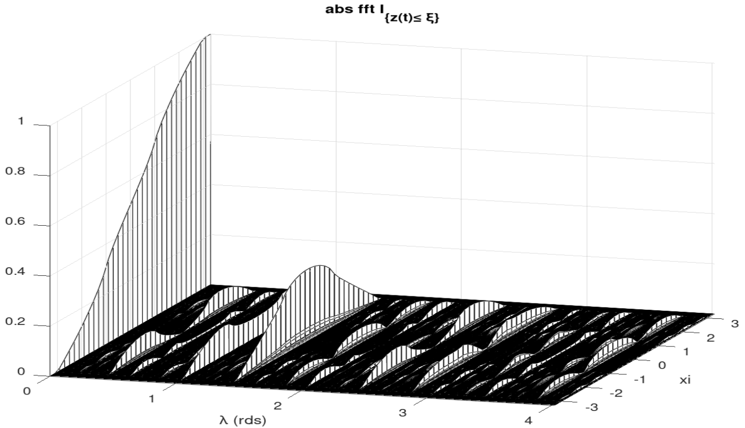
Let $z(t) = z_{\tau_1}(t) + z_{\tau_2}(t)$

where $z_{\tau_i}(t)$ bounded periodic function with period τ_i, $i = 1, 2$.

$\tau_1 > 0$ and $\tau_2 > 0$ uncommensurable: $\frac{\tau_1}{\tau_2} \notin \mathbb{Q}$ (not rational).

What kind of almost-periodicity does $\mathbb{I}_{\{z(t) \leq \xi\}}$ inherit ?

$$z(t) = \cos(2\pi t) - 2\cos(2\sqrt{2}\pi t)$$



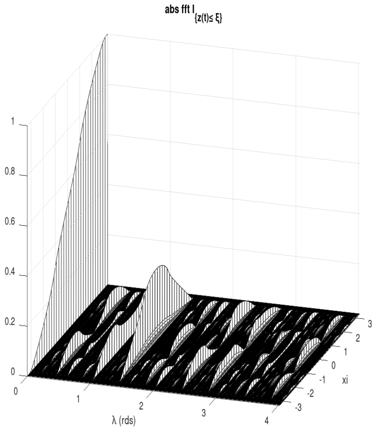
Apparition of **new harmonics**:

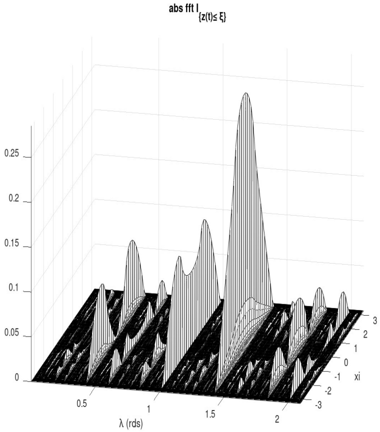
$$\gamma_k^{(1)} = k \qquad \text{and} \qquad \gamma_k^{(2)} = k\sqrt{2}$$

and also of **correlation between the frequencies** :

$$\gamma_{k_1,k_2} = k_1 + k_2\sqrt{2}, \quad k_1, k_2 \in \mathbb{Z} \quad !$$

abs fft $\mathbb{I}_{\{\cos(2\pi t)-2\cos(2\sqrt{2}\pi t)\leq\xi\}}$ $(\lambda \neq 0)$





Plan

I – **Almost periodicity** (a.p.)
  – Uniform (u.a.p.) – Stepanov (S-a.p.) – Besicovitch (B-a.p.)

II – **Indicator of an almost periodic function**

III – **Frequency extraction from the distribution function**
  – FOT distribution – Cyclic FOT-measure
  – Gardner fundamental theorem on sines-wave extraction

IV – B-**a.p.-in-distribution function**

V – **Almost periodic extraction**

VI – **Extraction of periodic components**

Besicovitch (1932): almost periodic functions in the sense of Bohr (uniform), Stepanoff, Weyl, and Besicovitch.
From M. Kac & H. Steinhaus (1937), M. Steinhaus (1940), notion of "relative distribution" reconsidered by W. Gardner (1987), J. Leśkow & A. Napolitano (2006): (with the notion of FOT-distribution)

I – Almost periodicity (a.p.)

– Uniform norm: $\mathcal{U}$: $N_{\mathcal{U}}(z) = \|z\|_\infty \triangleq \sup_t |z(t)| < \infty$;

– Stepanov S_T^p-norm:

$$N_{S_T^p}(z) = \|z\|_{S_T^p} \triangleq \sup_{t_o} \left[\frac{1}{T} \int_{t_o}^{t_o+T} |z(t)|^p \, dt \right]^{1/p};$$

– Besicovitch B^p-seminorm:

$$N_{B^p}(z) \triangleq \limsup_{T\to\infty} \left[\frac{1}{2T} \int_{-T}^{+T} |z(t)|^p \, dt \right]^{1/p}.$$

Besicovitch (1932): almost periodic functions in the sense of Bohr (uniform), of Stepanoff, of Weyl, and of Besicovitch.

Identification – Point separation property

(i) $N_u[z] = 0 \Leftrightarrow \big(z(t) = 0 \quad \text{for any } t\big)$.

(ii) $N_{S^p_T}[z] = 0 \Leftrightarrow \big(\, z(t) = 0 \quad \text{for Leb-almost every } t\,\big)$.

(iii) $N_{B^p}[z] = 0$: We can have $\mathrm{Leb}\{t \in \mathbb{R} : z(t) \neq 0\} = \infty$.
Examples: $(1 + |t|)^{-a}$ with $a > 0$, $e^{-|t|}, \ldots$
bounded relatively measurable $z(t)$ with Dirac FOT-distribution.

Some comparison properties

(i) $(1+T)^{-1} N_{S^p_1} \leq N_{S^p_T} \leq \left(1 + T^{-1}\right) N_{S^p_1}$. Notation : $S^p \triangleq S^p_1$

(ii) $N_{B^p} \leq N_{S^p_T} \leq N_{\mathcal{U}}$.

(iii) $N_{S^q} \leq N_{S^p}$ and $N_{B^q} \leq N_{B^p}$ for $1 \leq q < p$.

Almost periodic functions

Let $\mathcal{T}$ be the set of trigonometric polynomials.

– $\{\text{u.a.p}\} \triangleq \mathcal{C}_{\mathcal{U}}(\mathcal{T})$ (closure of $\mathcal{T}$ by the norm $N_{\mathcal{U}}$):

– $\{S^p\text{-a.p.}\} \triangleq \mathcal{C}_{S^p}(\mathcal{T})$ (closure of $\mathcal{T}$ by the norm N_{S^p});

– $\{B^p\text{-a.p.}\} \triangleq \mathcal{C}_{B^p}(\mathcal{T})$ (closure of $\mathcal{T}$ by the semi-norm N_{B^p}).

Properties Here $G = S$ or B.

(i) If $z_n(t)$ G^p-a.p. and $N_{G^p}(z_n - z) \to 0$, then $z(t)$ G^p-a.p.

(ii) If $z(t)$ u.a.p. then $z(t)$ bounded and uniformly continuous.

(iii) If $z(t)$ G^p-a.p. then $N_{G^p}[z] < \infty$.

(iv) $\{\text{u.a.p.}\} \subset \{S^p\text{-a.p.}\} \subset \{B^p\text{-a.p.}\}$.

(v) $\{G^p\text{-a.p.}\} \subset \{G^q\text{-a.p.}\} \subset \{G^1\text{-a.p.}\}$ for any $1 \leq q < p$.

Fourier analysis

Let $z(t)$ be G^p-a.p. Then

(i) The mean $a_\lambda^z \triangleq \left\langle z(t)\, e^{-j2\pi\lambda t} \right\rangle_t$ exists for any $\lambda \in \mathbb{R}$.

(ii) If $a_\lambda^z = 0$ for any $\lambda \in \mathbb{R}$, then $N_{G^p}[z] = 0$.

(iii) Let $\Gamma_z \triangleq \left\{\gamma \in \mathbb{R} : a_\gamma^z \neq 0\right\}$ and

$$\Lambda_z \triangleq \left\{ \sum_{i=1}^{n} n_i \gamma_i : n \in \mathbb{N}, n_i \in \mathbb{Z}, \gamma_i \in \Gamma_z, i = 1, \ldots, n \right\}.$$

Then the sets $\Gamma_z \subset \Lambda_z$ are at most countable.

(iv) Bochner–Fejér polynomial $\sigma_B^z(t)$ associated to a.p. $z(t)$:

$$\sigma_B^z(t) = \sum_{\gamma \in \Gamma_z \cap B} \alpha_{B,\gamma}^z a_\gamma^z\, e^{j2\pi\gamma t},$$

where $0 \leq \alpha_{B,\gamma}^z \leq 1$ and $B \subset \mathbb{R}$ finite.

(v) $\Gamma_{\sigma_B^z} \subset \Gamma_z$ and $\Gamma_{(\sigma_B^z)^k} \subset \Lambda_z$ for any $k \geq 1$.

(vi) $N_{G^p}\left[\sigma_{B_n}^z - z\right] \longrightarrow 0$ for any $B_n \uparrow \Gamma_z$ as $n \to \infty$.

(vii) Parseval equality:

$$\text{If} \quad z(t)\ B^2\text{-a.p.} \quad \underline{\text{then}} \quad \left\langle |z(t)|^2 \right\rangle_t = \sum_{\gamma \in \Gamma_z} |a_\gamma^z|^2 < \infty.$$

(viii) Riesz–Fisher theorem:

For every series $\sum_n a_n\, e^{j2\pi\gamma_n t}$ such that $\sum_n |a_n|^2 < \infty$,

<u>there exists</u> a B^2-a.p. $z(t)$ having this series as its Fourier series.

<u>Remark</u>: $z_1(t) = \cos(t)$ and $z_2(t) = z_1(t) + (1 + |t|)^{-1}$.
$N_{B^2}[z_1 - z_2] = 0$ and $z_1(t) \neq z_2(t)$ for any $t \in \mathbb{R}$.

Properties

(i) If $z_1(t)$ and $z_2(t)$ bounded G^1-a.p., <u>then</u> $z_1(t) \cdot z_2(t)$ G^1-a.p. Moreover $\Gamma_{z_1 \cdot z_2} \subset \Gamma_{z_1} + \Gamma_{z_2} \triangleq \{\gamma_1 + \gamma_2 : \gamma_2 \in \Gamma_{z_1}, \gamma_2 \in \Gamma_{z_2}\}$.

(ii) If $z(t)$ bounded G^1-a.p., <u>then</u> $z(t)^k$ G^p-a.p. for any $p \geq 1$ and any integer $k \geq 1$. Moreover $\Gamma_{z^k} \subset \Lambda_z$.

(iii) **If $z(t)$ bounded and G^1-a.p., and $g(x)$ continuous on $\mathbb{R}$**

<u>then</u> $g_o z : t \mapsto g(z(t))$ **is G^p-a.p. for any $p \geq 1$.**

Moreover $\Gamma_{g_o z} \subset \Lambda_z$**.**

These facts are well known for u.a.p. functions.

Convergence of the trigonometric series

If $z(t)$ is B^2-a.p. Then $\left\langle |z(t)|^2 \right\rangle_t = \sum_\lambda \left| a_z^\lambda \right|^2 < \infty$ (Parseval).
What can we say conversely ?

Consider a trigonometric series $\Sigma(t) \sim \sum_n a_n \, e^{j 2\pi \gamma_n t}$

(i) B^2-a.p.: If $\sum_n |a_n|^2 < \infty$ (Riesz–Fisher)

then there exists B^2-a.p. $z(t)$ with Fourier series $\Sigma(t)$.

This does not mean that series $\Sigma(t)$ is convergent for N_{B^2}.

But sequences of Bochner–Fisher polyn. $\{\sigma^z_{B_n}(t)\}$ converge to $z(t)$ for N_{B^2} (asymptotic). Also to $z(t) + 1/(1+|t|)$.

(ii) u.a.p: If $\sum_n |a_n| < \infty$ then $\Sigma(t)$ converges for any t, and is u.a.p.

(iii) S^2-a.p: if $\sum_n |a_n|^2 < \infty$ and $\sum\sum_{m \neq n} |a_n \, a_m| \frac{|\sin(\pi(\gamma_n - \gamma_m))|}{|\gamma_n - \gamma_m|} < \infty$,

then $\Sigma(t)$ converges in S^2, and is S^2-a.p.

II– Indicator of an almost periodic function

Relatively measurable function - FOT-distribution

Let $z(t)$ be a relatively measurable (RM) function
and $F_z(\xi)$ be its Fraction-Of-Time-distribution (FOT-distrib.).

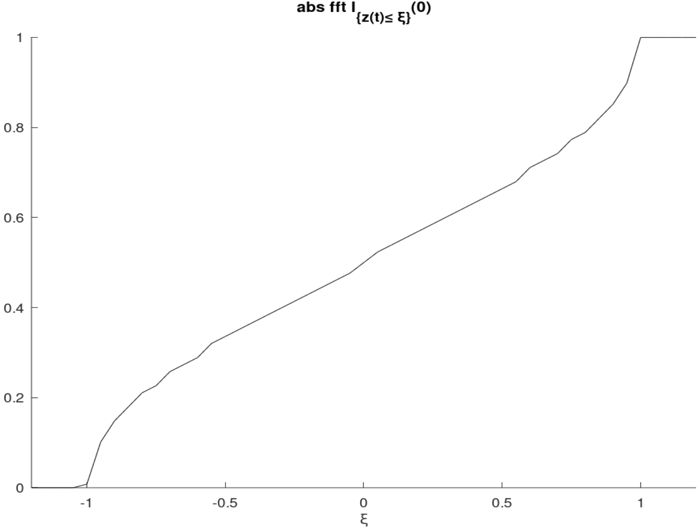
$$F_z(\xi) \triangleq \left\langle \mathbb{I}_{\{z(t)\le\xi\}} \right\rangle_t = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathbb{I}_{\{z(t)\le\xi\}} dt.$$

Wintner 1932: If $z(t)$ continuous bounded and $\left\langle x(t)^p \right\rangle_t$ exists for any $p \geq 1$ then $z(t)$ RM and

$$\left\langle x(t)^p \right\rangle_t = \int_{\mathbb{R}} \xi^p dF(\xi).$$

Hence, any u.a.p. function is RM.
As well as, any S^1-a.p. or B^1-a.p. bounded continuous function.

FOT distribution of $\cos(2\pi t)$: $F_{\cos(2\pi t)}(\xi)$



B^p-approximation of indicator function (Technical result)

For each $\epsilon > 0$, let $g_\epsilon(x)$ such that $\sup_x \left|\mathbb{I}_{\{x\geq 0\}} - g_\epsilon(x)\right| \leq 1$

$$\lim_{\epsilon\to 0} g_\epsilon(x) = \mathbb{I}_{\{x\geq 0\}} \quad \text{and} \quad \lim_{\epsilon\to 0} \sup_{|x|>\epsilon} \left|\mathbb{I}_{\{x\geq 0\}} - g_\epsilon(x)\right| = 0.$$

Let $z(t)$ RM function and 0 continuity point of $F_z(\xi)$.

Then $N_{B^p}\left[\mathbb{I}_{\{z(t)\geq 0\}} - g_{\epsilon_n}(z(t)\right] \to 0$

for $p \geq 1$ and for $\epsilon_n \to 0$ of continuity points of $F_z(\xi)$.

Remarks:

– Recall that $N_{B^p}(z) \triangleq \limsup_{T\to\infty}\left[\frac{1}{2T}\int_{-T}^{T} |z(t)|^p\, dt\right]^{1/p}$.

– Technical problem with other norms $N_{\mathcal{U}}$ and N_{S^p}.

B^p-a.p. indicator function

If $z(t)$ bounded, RM and B^1-a.p.
Then $\mathbb{I}_{\{z(t)\le\xi_o\}}$ B^p-a.p. for any $p \ge 1$, and $\Gamma_{\mathbb{I}_{\{z(t)\le\xi_o\}}} \subset \Lambda_z$ for any continuity point ξ_o of the FOT-distribution $F_z(\xi)$.

Remarks:

(i) Same property for $\mathbb{I}_{\{z(t)<\xi_o\}}$, $\mathbb{I}_{\{z(t)\ge\xi_o\}}$and $\mathbb{I}_{\{z(t)>\xi_o\}}$.

(ii) If $z(t)$ B^1-a.p. bounded then $\mathbb{I}_{\{z(t)\le\xi_o\}}$ B^1-a.p. for any $\xi_o \notin \Xi_z$ where Ξ_z at most countable.

(iii) We can also consider unbounded B^1-a.p. $z(t)$. Unfortunately we do not get the inclusion between $\Gamma_{\mathbb{I}_{\{z(t)\le\xi_o\}}}$ and Λ_z.

(iv) For a S^1-a.p. function we cannot conclude that $\mathbb{I}_{\{z(t)\le\xi_o\}}$ is S^1-a.p. for all points $\xi \in \mathbb{R}$ except an at most countable subset.
But only for some points.

III– Frequency extraction

Cyclic FOT-measure

Let $\Lambda \subset \mathbb{R}$ such that: if $\lambda \in \Lambda$ then $k\lambda \in \Lambda$ for any $k \in \mathbb{Z}$.

Definition $z(t) \in \widetilde{\mathcal{Z}}_b^{(\Lambda)}$:

$$\sup_t |z(t)| < \infty \quad \text{and} \quad F_z^\lambda(\xi) \triangleq \left\langle \mathbb{I}_{\{z(t)\leq\xi\}}\, e^{-j2\pi\lambda t} \right\rangle_t \quad \text{exists}$$

for any $\lambda \in \Lambda$ and $\xi \in \mathbb{R} \setminus \Xi_z$ where $\Xi_z \subset \mathbb{R}$ is at most countable.

Then

(i) $z(t)$ RM, $F_z(\xi) \triangleq F_z^0(\xi)$ FOT-distribution of $z(t)$.

(ii) For $\lambda \in \Lambda$, $\xi \in \mathbb{R} \setminus \Xi_z$,

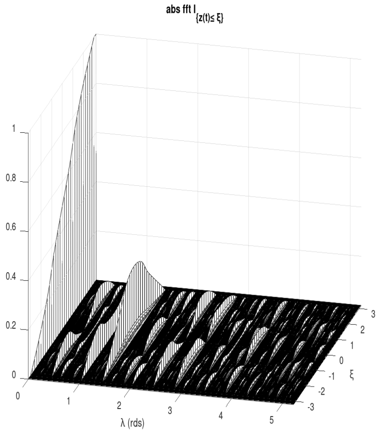
$$F_z^\lambda(\xi) \in \mathbb{C}, \quad F_z^{-\lambda}(\xi) = \overline{F_z^\lambda(\xi)} \quad \text{and} \quad \left|F_z^\lambda(\xi)\right| \leq F_z(\xi) \leq 1,$$

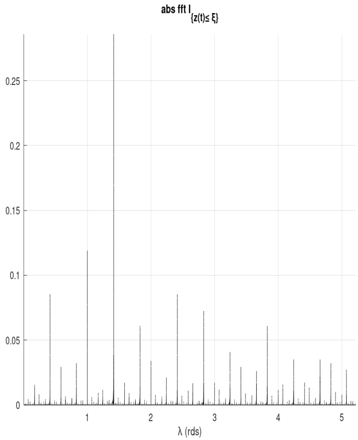
$$F_z^\lambda(-\infty) = 0 \text{ for any } \lambda, \quad \text{and} \quad F_z^\lambda(\infty) = 0 \text{ for any } \lambda \neq 0.$$

(iii) $\left|F_z^\lambda(\xi_2) - F_z^\lambda(\xi_1)\right| \leq F_z(\xi_2) - F_z(\xi_1)$ for $\xi_1 \leq \xi_2$ in $\mathbb{R} \setminus \Xi_z$.

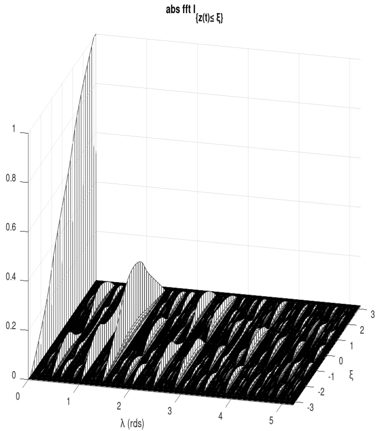
Recall $F_z(-\xi)$ non-decreasing and (since $\sup_t |z(t)| < \infty$)
$F_z(-\xi) \longrightarrow F_z(-\infty) = 0$ $\quad F_z(\xi) \longrightarrow F_z(\infty) = 1$ as $\xi \to \infty$.

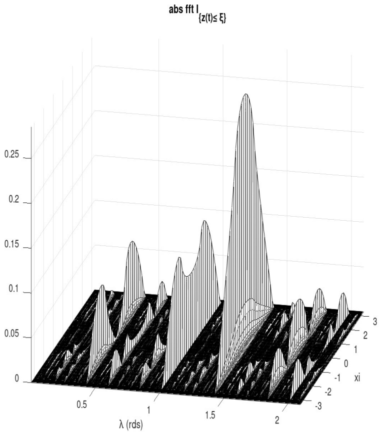
abs fft $\mathbb{I}_{\{\cos(2\pi t)-2\cos(2\sqrt{2}\pi t)\leq\xi\}}$ $\lambda \neq 0,$ $\max_\xi$ abs fft





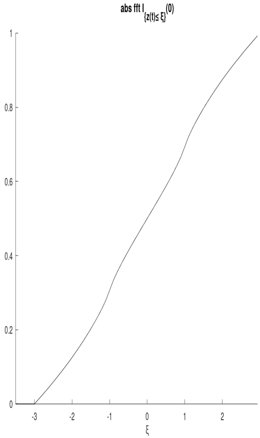
abs fft $\mathbb{I}_{\{\cos(2\pi t)-2\cos(2\sqrt{2}\pi t)\leq\xi\}}$ $\lambda \neq 0$ zoom



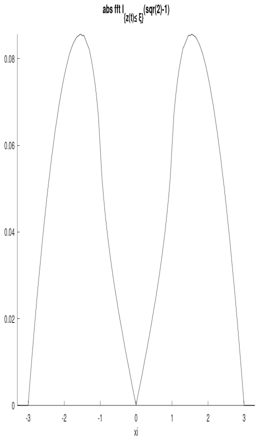


abs fft $\mathbb{I}_{\{\cos(2\pi t)-2\cos(2\sqrt{2}\pi t)\leq\xi\}}$

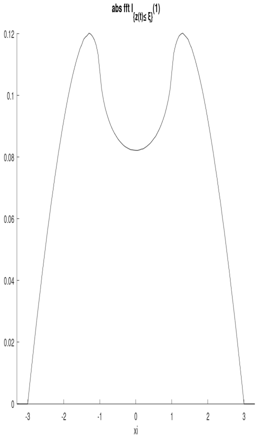
$\lambda = 0$



$\lambda = \sqrt{2} - 1 \approx 0.4$

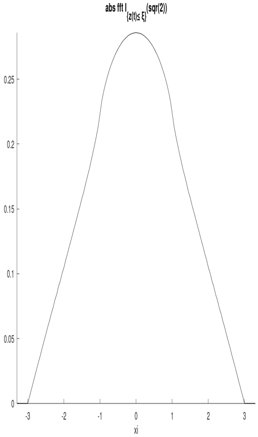


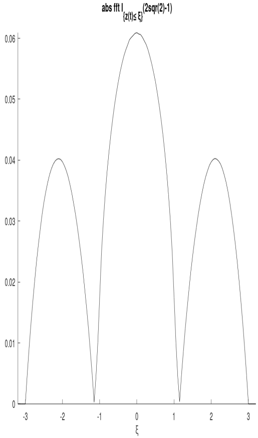
$\lambda = 1$

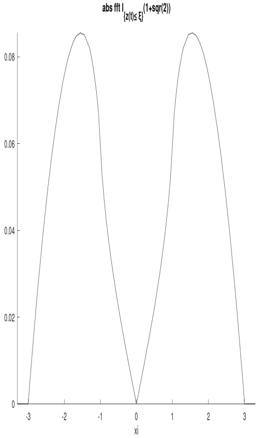


abs fft $\mathbb{I}_{\{\cos(2\pi t)-2\cos(2\sqrt{2}\pi t)\leq\xi\}}$

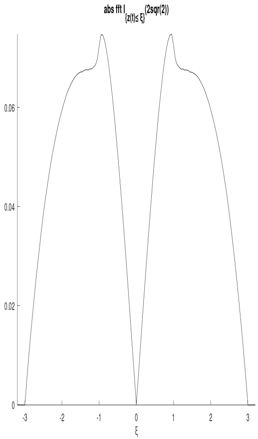
$\lambda=\sqrt{2}\approx 1.4$ $\qquad$ $\lambda=2\sqrt{2}-1\approx 1.8$ $\qquad$ $\lambda=\sqrt{2}+1\approx 2,4$



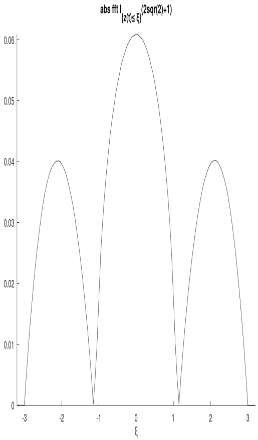




$\lambda = 2\sqrt{2} \approx 2.8$



$\lambda = 2\sqrt{2} + 1 \approx 3.8$



Let $z(t) \in \widetilde{\mathcal{Z}}_b^{(\Lambda)}$ and $\lambda \in \Lambda$. We have seen that

$$\left|F_z^\lambda(\xi_2) - F_z^\lambda(\xi_1)\right| \leq F_z(\xi_2) - F_z(\xi_1) \quad \text{for } \xi_1 \leq \xi_2 \text{ in } \mathbb{R} \setminus \Xi_z .$$

The increments of $F_z^\lambda(\xi)$ are dominated by the increments of the FOT-distribution $F_z(\xi)$ of $z(t)$.

Hence $\xi \mapsto F_z^\lambda(\xi)$ is of bounded variation in $\mathbb{R}$ and is continuous at any point of continuity of the FOT-distribution $F_z(\xi)$.

Definition

The function $F_z^\lambda(\xi)$ will be called **cyclic FOT-measure** at frequency λ of the function $z(t)$.

Stieltjes integral

Fundamental theorem on sines-wave extraction (W. Gardner)

Let $z(t) \in \widetilde{\mathcal{Z}}_b^{(\Lambda)}$ and $g(\xi)$ be a function which is
(1) either continuous,
(2) or bounded, monotonic and $\int_{\mathbb{R}} |g(\xi)| dF_z(\xi)$ exists.
Then

$$\left\langle g(z(t))\, e^{-j2\pi\lambda t} \right\rangle_t \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{t_o - T}^{T+t_o} g(z(t))\, e^{-j2\pi\lambda t}\, dt$$

exists independently of $t_o \in \mathbb{R}$, and

$$\left\langle g(z(t))\, e^{-j2\pi\lambda t} \right\rangle_t = \int_{\mathbb{R}} g(\xi)\, dF_z^{\lambda}(\xi),$$

for any $\lambda \in \Lambda$.

Hence, for every $\lambda \in \Lambda$ and integer $k \geq 1$,

$$\left\langle z(t)^k e^{-j2\pi\lambda t} \right\rangle_t = \int_{\mathbb{R}} \xi^k\, dF_z^{\lambda}(\xi).$$

IV– B-a.p.-in-distribution function

Definition

$z(t) \in \widetilde{\mathcal{Z}}_b^{ap}$: $z(t)$ bounded and B^1**-a.p. in distribution**:
$\mathbb{I}_{\{z(t) \leq \xi\}}$ is B^1-a.p. for any $\xi \in \mathbb{R} \setminus \Xi_z$, where $\Xi_z \subset \mathbb{R}$ at most countable.

Then

(i) $z(t) \in \widetilde{\mathcal{Z}}_b^{(\mathbb{R})}$, $\quad F_z^\lambda(\xi) \triangleq \left\langle \mathbb{I}_{\{z(t) \leq \xi\}} \, e^{j2\pi\lambda t} \right\rangle_t$ and

$\widetilde{\Gamma}_{z,\xi} \triangleq \left\{ \lambda \in \mathbb{R} : F_z^\lambda(\xi) \neq 0 \right\}$ at most countable for any $\xi \notin \Xi_z$.

Let $\widetilde{\Gamma}_z \triangleq \bigcup_{\xi \notin \Xi_z} \widetilde{\Gamma}_{z,\xi}$.

(ii) Furthermore

$$a_z^\lambda \triangleq \left\langle z(t) e^{-j2\pi\lambda t} \right\rangle_t = \int_{\mathbb{R}} \xi \, dF_z^\lambda(\xi)$$

is well-defined for any $\lambda \in \mathbb{R}$.

(iii) Hence $z(t) \in \mathcal{Z}_b^{(\mathbb{R})}$ and $\Gamma_z \subset \widetilde{\Gamma}_z$.

(iv) Parseval equality :

$$F_z(\xi) = \left\langle \left(\mathbb{I}_{\{z(t)\leq\xi\}}\right)^2 \right\rangle_t = \sum_{\lambda\in\widetilde{\Gamma}_z} \left|F_z^\lambda(\xi)\right|^2 \leq 1.$$

Hence

$$\sum_{\lambda\in\widetilde{\Gamma}_z\setminus\{0\}} \left|F_z^\lambda(\xi)\right|^2 = F_z(\xi)\big(1-F_z(\xi)\big) \leq \min\left\{1/4\,,\ F_z(\xi)\,,\ 1-F_z(\xi)\right\}.$$

As result

– If $F_z^\lambda(\xi) = 0$ for any $\lambda \neq 0$ <u>then</u> $F_z(\xi) = 0$ or 1.

– If $\widetilde{\Gamma}_z = \{0\}$ <u>then</u> there exists $\xi_o \in \mathbb{R}$ such that $N_B[z(t) - \xi_o] = 0$.

(v) For the increments (Parseval equality):

$$\sum_{\lambda\in\widetilde{\Gamma}_z} \left|F_z^\lambda(\xi_2) - F_z^\lambda(\xi_1)\right|^2 = \left\langle \mathbb{I}_{\{\xi_1 < z(t) \le \xi_2\}} \right\rangle_t = F_z(\xi_2) - F_z(\xi_1) \le 1$$

and

$$\sum_{\lambda\in\widetilde{\Gamma}_z\setminus\{0\}} \left|F_z^\lambda(\xi_2) - F_z^\lambda(\xi_1)\right|^2 = \left(F_z(\xi_2) - F_z(\xi_1)\right)\left(1 - F_z(\xi_2) + F_z(\xi_1)\right)$$

$$\le \min\{1/4\,,\ \left(F_z(\xi_2) - F_z(\xi_1)\right),\ \left(1 - F_z(\xi_2) + F_z(\xi_1)\right)\}$$

for $\xi_1 \le \xi_2$ in $\mathbb{R} \setminus \Xi_z$.

(v) However when $z(t) \in \widetilde{\mathcal{Z}}_b^{ap}$ we do not know whether $z(t)$ is B-a.p.
Even when $\sum_\lambda \left|a_z^\lambda\right|^2 < \infty$ we do not now whether $z(t)$ is B^2-a.p.

Case of a bounded and B-a.p. function

Let $z(t)$ bounded and B^1-a.p.

Then $z(t)$ B^2-a.p. and $\sum_\lambda \left|a_z^\lambda\right|^2 < \infty$.

Moreover $z(t) \in \widetilde{\mathcal{Z}}_b^{ap} \subset \widetilde{\mathcal{Z}}^{(\mathbb{R})}$ and $\widetilde{\Gamma}_z \subset \Lambda_z$.

Recall $\Lambda_z \triangleq \left\{ \sum_{i=1}^n n_i \gamma_i : n \in \mathbb{N}, n_i \in \mathbb{Z}, \gamma_i \in \Gamma_z, i = 1, \dots, n \right\}$.

The previous results are valid.

V – Almost periodic extraction

We consider two ways to extract an almost periodic part of a signal.

- The first one is directly characterized by the Fourier (or cyclic) coefficients of the signal (the almost periodic additive component). It fits very well for linear analysis.
- The second one is defined from the cyclic FOT measures (the almost periodic FOT-distribution component). It can be applied for non linear analysis following Gardner fundamental theorem of sines-wave extraction.

Unfortunately the relationships between these two notions are not satisfactory. Then we illustrate this problem in the case of the periodic extraction.

Almost periodic additive component

Let $z(t) \in \mathcal{Z}^{(\Lambda)}$, that is, $a_z^\lambda \triangleq \left\langle z(t) e^{-j2\pi\lambda t} \right\rangle_t$ defined for $\lambda \in \Lambda$.
Assume that

$$\sum_{\lambda \in \Lambda} \left| a_z^\lambda \right|^2 < \infty.$$

Then there exists a (not unique) B^2-a.p. function $z_\Lambda(t)$ with

$$z_\Lambda(t) \sim \sum_{\lambda \in \Lambda} a_z^\lambda \, e^{j2\pi\lambda t} \qquad \text{(Riesz–Fisher theorem)}$$

Hence $a_{z_\Lambda}^\lambda \triangleq \left\langle z_\Lambda(t) \, e^{-j2\pi\lambda t} \right\rangle_t = a_z^\lambda$ for $\lambda \in \Lambda$

and $a_{z_\Lambda}^\lambda = 0$ for $\lambda \in \mathbb{R} \setminus \Lambda$.

Moreover

$$\left\langle \left| z_\Lambda(t) \right|^2 \right\rangle_t = \sum_{\lambda \in \Lambda} \left| a_z^\lambda \right|^2 \quad \text{(Parseval inequality).}$$

Define the residual $z_{\Lambda,r}(t) \triangleq z(t) - z_\Lambda(t)$.

Then $\left\langle z_{\Lambda,r}(t) e^{-j2\pi\lambda t} \right\rangle_t = 0$ for any $\lambda \in \Lambda$.

and $\left\langle z_{\Lambda,r}(t)\, z_\Lambda(t)^* \right\rangle_t = 0$ when either (i) $z(t)$ and $z_\Lambda(t)$ bounded,
or (ii) $\langle z(t)^2 \rangle_t$ exists and is finite.

In the case (ii) we obtain that

$$\langle |z_{\Lambda,r}(t)|^2 \rangle_t = \langle |z(t)|^2 \rangle_t - \langle |z_\Lambda(t)|^2 \rangle_t .$$

Notice that $z_\Lambda(t)$ is real-valued if $\Lambda = -\Lambda$.

* If $\sum_{\lambda \in \Lambda} \left| a_z^\lambda \right| < \infty$ then $z_\Lambda(t)$ is uniformly almost periodic (u.a.p.)

$$z_\Lambda(t) = \sum_{\lambda \in \Lambda} a_z^\lambda \, e^{j2\pi\lambda t} .$$

The sum is uniform with respect to $t \in \mathbb{R}$;

and $z_\Lambda(t)$ continuous bounded.

Almost periodic-distribution component

Let $z(t) \in \widetilde{\mathscr{Z}}^{(\Lambda)}$ and $\Lambda \subset \mathbb{R}$ stable by integer multiplication.

Then cyclic FOT-measure $F_z^\lambda(\xi) \triangleq \left\langle \mathbb{I}_{\{z(t) \leq \xi\}} \, e^{-j2\pi\lambda t} \right\rangle_t$ for $\lambda \in \Lambda$.

If in addition $\sum_{\lambda \in \Lambda} \left| F_z^\lambda(\xi) \right|^2 < \infty$ for $\xi \in \mathbb{R} \setminus \Xi_z$,

then there exists B^2-a.p. $t \mapsto \Phi_z^{(\Lambda)}(t, \xi) \in \mathbb{R}$ with

$$\Phi_z^{(\Lambda)}(t, \xi) \sim \sum_{\lambda \in \Lambda} F_z^\lambda(\xi) \, e^{j2\pi\lambda t} \qquad \text{(Riesz–Fisher theorem)}$$

and

$$\left\langle \Phi_z^{(\Lambda)}(t, \xi)^2 \right\rangle_t = \sum_{\lambda \in \Lambda} \left| F_z^\lambda(\xi) \right|^2 \qquad \text{(Parseval equality).}$$

Let the residual $R_z^{(\Lambda)}(t,\xi) \triangleq \mathbb{I}_{\{z(t)\leq\xi\}} - \Phi_z^{(\Lambda)}(t,\xi)$.

Then $\left\langle R_z^{(\Lambda)}(t,\xi)\, e^{-j2\pi\lambda t} \right\rangle_t = 0$ for any $\lambda \in \Lambda$,

$$\left\langle R_z^{\Lambda}(t,\xi)\, \Phi^{(\Lambda)}(t,\xi) \right\rangle_t = 0$$

and

$$\left\langle R_z^{(\Lambda)}(t,\xi)^2 \right\rangle_t = F_z(\xi) - \left\langle \Phi_z^{(\Lambda)}(t,\xi)^2 \right\rangle_t.$$

If $z(t) \in \widetilde{\mathcal{Z}}^{(\Lambda)}$ and bounded

$$\underline{\text{then}} \qquad z(t) \in \mathcal{Z}^{(\Lambda)} \quad \text{and} \quad a_z^\lambda \triangleq \left\langle z(t) e^{-j2\pi\lambda t} \right\rangle_t = \int_{\mathbb{R}} \xi \, dF_z^\lambda(\xi).$$

(i) If in addition $\sum_{\lambda\in\Lambda} \left|a_z^\lambda\right|^2 < \infty$ <u>then</u> there exits a B^2-a.p. $z_\Lambda(t)$

$$z_\Lambda(t) \sim \sum_{\lambda\in\Lambda} a_z^\lambda \, e^{j2\pi\lambda t}.$$

Moreover $\Gamma_{z_\Lambda} \subset \Lambda$, $a_{z_\Lambda}^\lambda = 0$ for $\lambda \notin \Lambda$ and

$$a_{z_\Lambda}^\lambda = a_z^\lambda = \int_{\mathbb{R}} \xi \, dF_z^\lambda(\xi) \quad \text{for } \lambda \in \Lambda.$$

(ii) Let $z_{\Lambda,r}(t) \triangleq z(t) - z_\Lambda(t)$.
<u>Then</u> $\left\langle z_{\Lambda,r}(t) e^{-j2\pi\lambda t} \right\rangle_t = 0$ for any $\lambda \in \Lambda$.

(iii) If we also assume that the B^2-a.p. function $z_\Lambda(t)$ is bounded,

then $\quad z_\Lambda(t) \in \widetilde{\mathcal{Z}}^{(\Lambda_{z_\Lambda})}$.

Furthermore $a^\lambda_{z_\Lambda} = 0$ for $\lambda \notin \Lambda$ and

$$a^\lambda_{z_\Lambda} = a^\lambda_z = \int_{\mathbb{R}} \xi \, dF^\lambda_z(\xi) = \int_{\mathbb{R}} \xi \, dF^\lambda_{z_\Lambda}(\xi) \quad \text{for} \quad \lambda \in \Lambda.$$

Question: What is the link between $F^\lambda_{z_\Lambda}(\xi)$ and $F^\lambda_z(\xi)$?

VII–Extraction of periodic components

Periodic additif component of a signal

Definition Let $\tau_1 > 0$ fixed, $\gamma_1 \triangleq \tau_1^{-1}$ and $\Lambda_1 \triangleq \gamma_1 \mathbb{Z}$.

$z(t) \in \mathcal{Z}_b^{\{\tau_1\}}$: $z(t)$ bounded and synchronized average

$$z_{\tau_1}(t) \triangleq \mathrm{E}^{\{\tau_1\}}\{z(t)\} \triangleq \lim_{N\to\infty} \frac{1}{2N} \sum_{n=-N}^{N-1} z(t+n\tau_1) = \langle z(t+n\tau_1) \rangle_n$$

exists for any $t \in \mathcal{R} \subset \mathbb{R}$ where $\mathrm{Leb}[\mathbb{R} \setminus \mathcal{R}] = 0$.

Then z_{τ_1} periodic bounded and $z_{\tau_1}(t) \in \mathcal{Z}^{(\gamma_1 \mathbb{Z})}$.

Moreover $a_{z_{\tau_1}}^{\lambda} = 0$ for $\lambda \notin \gamma_1 \mathbb{Z}$ and

$$a_{z_{\tau_1}}^{k\gamma_1} \triangleq \left\langle z_{\tau_1}(t) e^{-j2\pi k\gamma_1 t} \right\rangle_t = \frac{1}{\tau_1} \int_0^{\tau_1} z_{\tau_1}(t) e^{-j2\pi k\gamma_1 t}\, dt.$$

Let $z(t) \in \mathcal{Z}_b^{\{\tau_1\}} \cap \mathcal{Z}_b^{(\gamma_1 \mathbb{Z})}$.

Periodic extraction: $z(t) = z_{\tau_1}(t) + z_{\tau_1,r}(t)$.

Then residual $z_{\tau_1,r}(t) \stackrel{\Delta}{=} z(t) - z_{\tau_1}(t) \in \mathcal{Z}_b^{(\gamma_1 \mathbb{Z})}$ and

$$\langle z(t) e^{-j2\pi k\gamma_1 t} \rangle_t = \langle z_{\tau_1}(t) e^{-j2\pi k\gamma_1 t} \rangle_t,$$

$$\langle z_{\tau_1,r}(t) e^{-j2\pi k\gamma_1 t} \rangle_t = 0 \quad \text{for} \quad k \in \mathbb{Z}.$$

that is

$$a_z^{k\gamma_1} = a_{z_{\tau_1}}^{k\gamma_1} \quad \text{and} \quad a_{z_{\tau_1,r}}^{k\gamma_1} = 0.$$

If $\lambda \notin \gamma_1 \mathbb{Z}$ and $a_z^\lambda \stackrel{\Delta}{=} \langle z(t) e^{-j2\pi\lambda t} \rangle_t$ exists then

$$\langle z(t) e^{-j2\pi\lambda t} \rangle_t = \langle z_{\tau_1,r}(t) e^{-j2\pi\lambda t} \rangle_t \quad \text{and} \quad \langle z_{\tau_1}(t) e^{-j2\pi\lambda t} \rangle_t = 0$$

that is

$$a_z^\lambda = a_{z_{\tau_1,r}}^\lambda \quad \text{and} \quad a_{z_{\tau_1}}^\lambda = 0.$$

"Conversely" : If $z(t) \in \mathcal{Z}_b^{(\gamma_1 \mathbb{Z})}$, have we $z(t) \in \mathcal{Z}_b^{\tau_1}$?

Periodic distribution component of a signal

Definition Let $\tau_1 > 0$ fixed, $\gamma_1 \triangleq \tau_1^{-1}$ and $\Lambda_1 = \gamma_1 \mathbb{Z}$.

$z(t) \in \tilde{\mathcal{Z}}^{\{\tau_1\}}$: synchronized average

$$\Phi_z^{\{\tau_1\}}(t, \xi) \triangleq \mathrm{E}^{\{\tau_1\}}\left\{\mathbb{I}_{\{z(t) \leq \xi\}}\right\} = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \mathbb{I}_{\{z(t+n\tau_1) \leq \xi\}},$$

exists for any $t \in \mathcal{R} \subset \mathbb{R}$ and any $\xi \in \mathbb{R} \setminus \Xi$.

where $\mathrm{Leb}(\mathbb{R} \setminus \mathcal{R}) = 0$ and the set $\Xi \subset \mathbb{R}$ is at most a countable set. (Napolitano 2020, Definition 2.20 (p.46)).

For simplicity of presentation, put $\Phi_z^{\{\tau_1\}}(t, \xi) \triangleq 0$ for $t \in \mathbb{R} \setminus \mathcal{R}$.

We readily obtain that $z(t)$ is RM. Moreover

(i) $\xi \mapsto \Phi_z^{\{\tau_1\}}(t,\xi)$ is a FOT-distribution for $t \in \mathcal{R}$.

(ii) $t \mapsto \Phi_z^{\{\tau_1\}}(t,\xi)$ periodic and $0 \leq \Phi_z^{\{\tau_1\}}(t,\xi) \leq 1.$ for $\xi \in \mathbb{R} \setminus \Xi$.

(iii) Define

$$F_z^{\{\tau_1\}}(\lambda,\xi) \triangleq \left\langle \Phi_z^{\{\tau_1\}}(t,\xi)\, e^{-j2\pi\lambda t} \right\rangle_t \quad \text{for} \quad \lambda \in \mathbb{R}.$$

<u>Then</u> $F_z^{\{\tau_1\}}(\lambda,\xi) = 0$ for $\lambda \notin \gamma_1\mathbb{Z}$ and

$$F_z^{\{\tau_1\}}(\lambda,\xi) = \frac{1}{\tau_1}\int_0^{\tau_1} \Phi_z^{\{\tau_1\}}(t,\xi)\, e^{-j2\pi k\gamma_1 t}\, dt \quad \text{if} \quad k \in \mathbb{Z}.$$

(iv) $\tilde{\mathcal{Z}}^{\{\tau_1\}} \subset \tilde{\mathcal{Z}}^{(\gamma_1\mathbb{Z})}$ and

$$F_z^{\{\tau_1\}}(\lambda,\xi) = \left\langle \mathbb{I}_{\{z(t)\leq\xi\}}\, e^{-j2\pi\lambda t} \right\rangle_t \triangleq F_z^{\lambda}(\xi) \quad \text{for} \quad \lambda \in \gamma_1\mathbb{Z}.$$

Notice $F_z^{\{\tau_1\}}(0,\xi) = F_z(\xi)$.

Extraction of a finite number of periodic components

$\tau_1 > 0$ and $\tau_2 > 0$ non commensurable : $\frac{\tau_1}{\tau_2}$ irrational.

$\Lambda = \gamma_1 \mathbb{Z} \cup \gamma_2 \mathbb{Z}$.

(i) Additif component of a signal

$$z_{12}(t) \stackrel{\Delta}{=} z_{\tau_1}(t) + z_{\tau_2}(t) - \langle z(t) \rangle_t .$$

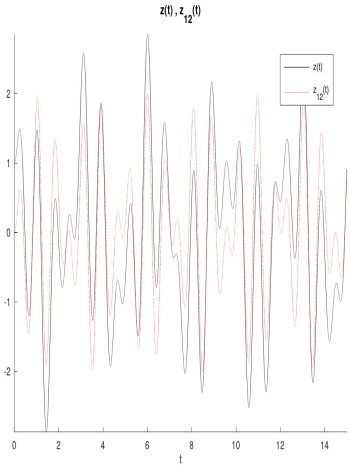
Recall $\left\langle z(t) \right\rangle_t = \left\langle z_{\tau_1}(t) \right\rangle_t = \left\langle z_{\tau_2}(t) \right\rangle_t$.

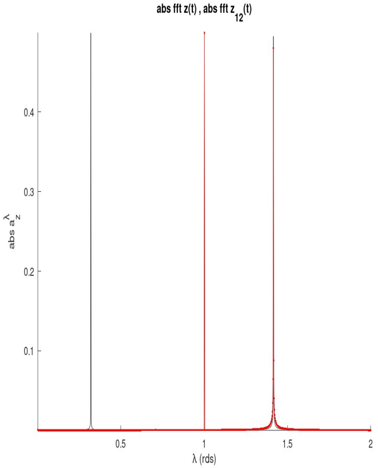
(ii) Distribution component of a signal

$$\Phi_z^{12}(t,\xi) \stackrel{\Delta}{=} \Phi_z^{\tau_1}(t,\xi) + \Phi_z^{\tau_2}(t,\xi) - F_z(\xi).$$

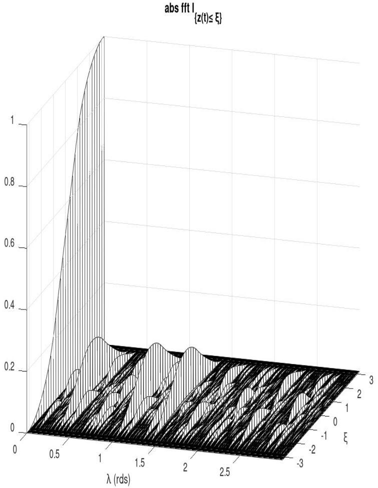
Recall $F_z(\xi) \stackrel{\Delta}{=} \left\langle \mathbb{I}_{\{z(t) \leq \xi\}} \right\rangle_t = \left\langle \Phi_z^{\tau_1}(t,\xi) \right\rangle_t = \left\langle \Phi_z^{\tau_2}(t,\xi) \right\rangle_t$

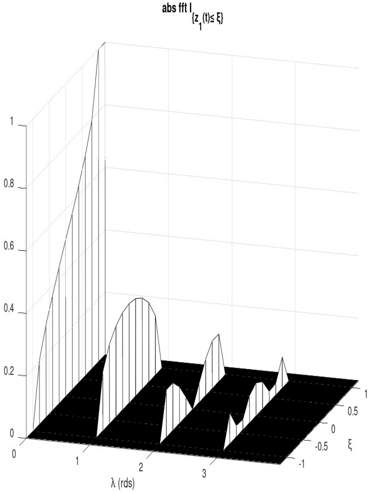
Simulation: ap-add.extract. $\cos(2\pi t) - \cos(2\sqrt{2}\pi t) + \cos(2t)$





freq.extract.indic. $\cos(2\pi t) - \cos(2\sqrt{2}\pi t) + \cos(2t)$





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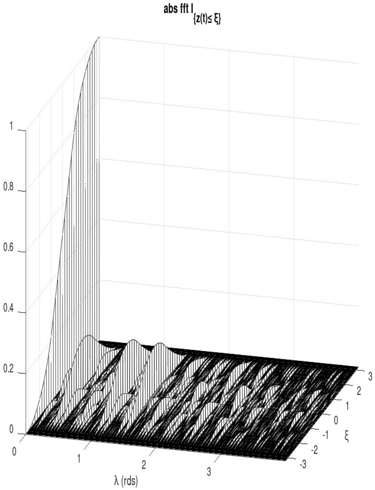

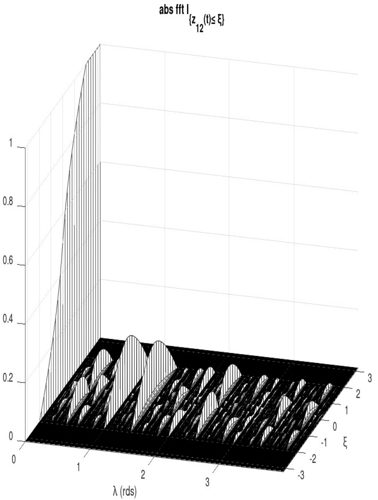

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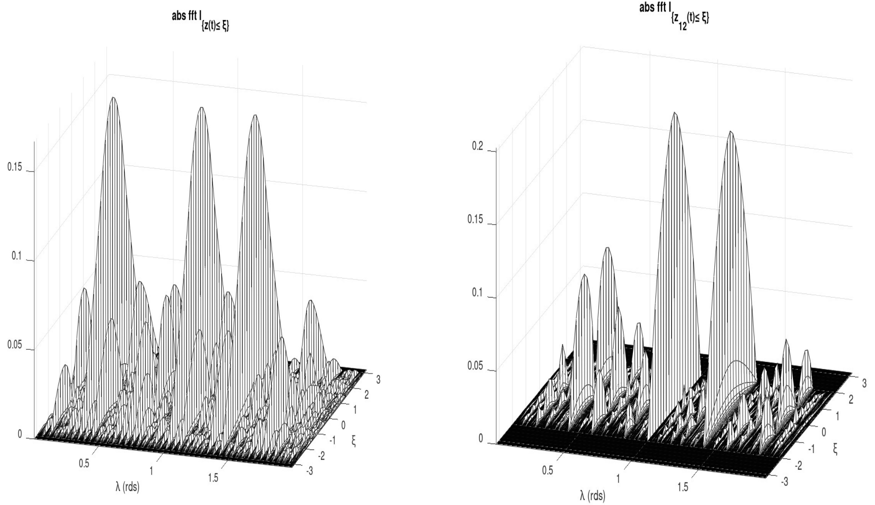
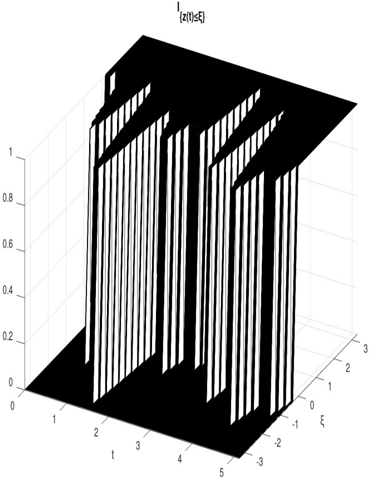
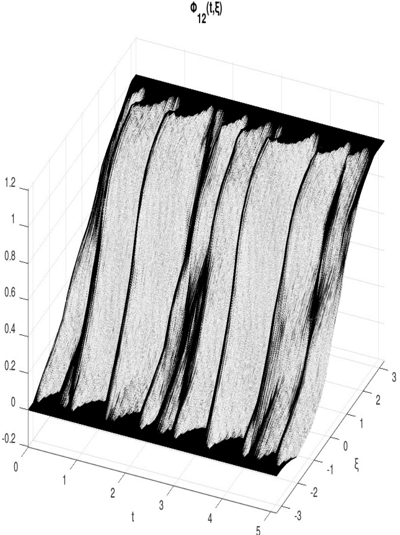
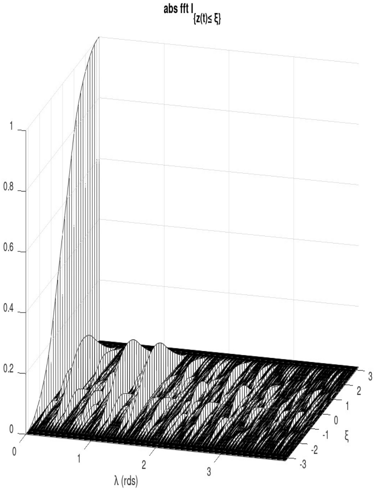


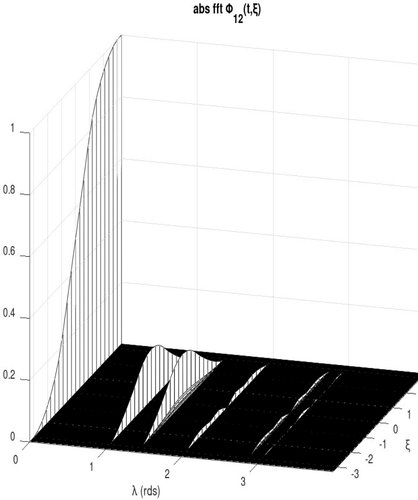
ap-distr.extract. $\cos(2\pi t) - \cos(2\sqrt{2}\pi t) + \cos(2t)$



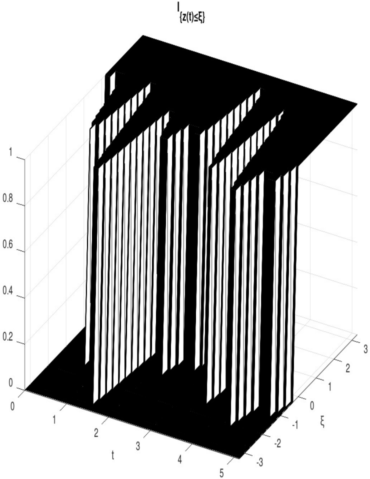


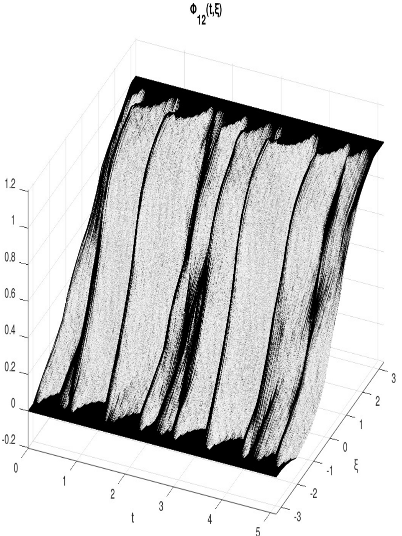
freq.extract. $\cos(2\pi t) - \cos(2\sqrt{2}\pi t) + \cos(2t)$





ap-distr.extract.: remark





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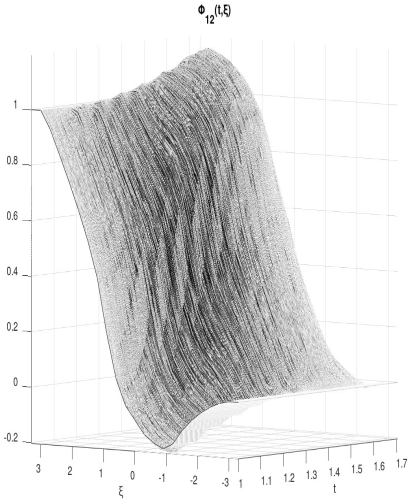

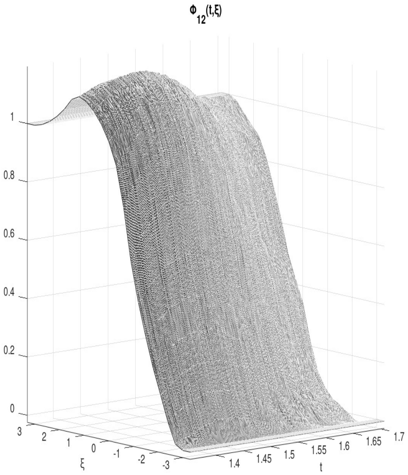

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THANK YOU FOR YOUR ATTENTION !