Generalized Spectral Coherence for cyclostationary signals with α-stable distribution

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HR EXCELLENCE IN RESEARCH

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Agenda

- Periodically correlated random sequences,
- α-stable distribution,
- α-stable cyclostationary random sequences,
- Generalised spectra coherence
- Application to simulated data

Stationary

Let us consider a random sequence $\{X_t\}$. It is called strictly stationary if for each $n \in \mathbb{Z}$, times $t_1, \dots, t_n \in \mathbb{Z}$ and Borel sets $A_1, \dots, A_n$ the following holds:

$$P_{t_{1+1},\dots,t_{n+1}}(A_1, \dots, A_n) = P\left(X_{t_{1+1}} \in A_1, \dots, X_{t_{n+1}} \in A_n\right) = P_{t_1,\dots,t_n}(A_1, \dots, A_n)$$

A second order random sequence $X_t \in L^2(\Omega, \mathcal{F}, P)$ with $t \in \mathbb{Z}$ is called weakly stationary, if for every $s, t \in \mathbb{Z}$:

$$m(t) = m, \qquad \mathrm{Cov}(X_s, X_t) = R(s - t)$$

Periodically Correlated

Let us consider a random sequence $\{X_t\}$. It is called strictly periodically correlated with period T if for each $n \in \mathbb{Z}$, times $t_1, \dots, t_n \in \mathbb{Z}$ and Borel sets $A_1, \dots, A_n$ the following holds:

$$P_{(t_{1+T},\dots,t_{n+T})}(A_1, \dots A_n) = P_{(t_1,\dots,t_n)}(A_1, \dots A_n)$$

A second order random sequence $\mathrm{X_t} \in \mathrm{L}^2(\Omega, \mathcal{F}, \mathrm{P})$ with $t \in \mathbb{Z}$ and period T is called periodically correlated, if for every $\mathrm{s}, t \in \mathbb{Z}$:

$$m(t) = m(t+T), \qquad \mathrm{Cov}(X_s, X_t) = \mathrm{Cov}(X_{s+T}, X_{t+T})$$

Generalized spectra coherenc

Let consider PC time series $\{X_t\}, t \in \mathbb{Z}$ with period $T \in \mathbb{N}$. Then spectral correlation at cycle frequency ϵ is defined as

$$S_X(f, \epsilon) = \sum_{\tau=-\infty}^{\infty} R_X^{\epsilon}(\tau) e^{-i2\pi f \tau}.$$

And Generalized spectral coherence is

$$|\gamma_X(f, \epsilon)|^2 = \lim_{N \to \infty} \frac{\left|S_X^{(N)}(f, \epsilon)\right|^2}{S_X^{(N)}(f + \epsilon/2, 0) S_X^{(N)}(f - \epsilon/2, 0)},$$

where

$$S_X^{(N)}(f, \epsilon) = \sum_{\tau=-\infty}^{\infty} \sum_{t=-N/2}^{N/2} R_X(t, \tau) e^{-i2\pi f \tau} e^{-i2\pi \epsilon t}.$$

Periodically Correlated – Example 1

Let $\{Z_t\}$ be the weakly stationary random sequence, where $t \in \mathbb{Z}$, $\mathbb{E}Z_t = m$ and $\mathrm{Var}(Z_t) = \sigma^2 < \infty$. Moreover, $f(t)$ $f: \mathbb{Z} \to \mathbb{R}$ is a periodic function with period T. Then, the random sequence $\{X_t\}$ defined as:

$$X_t = f(t) Z_t$$

is the second-order PC time series.

Periodically Correlated – Example 2

The second-order PARMA*(p,q)* time series is defined as follows

$$X_t - \phi_1(t)X_{t-1} - \cdots - \phi_p(t)X_{t-p} = \xi_t + \theta_1(t)\xi_{t-1} + \cdots + \theta_q(t)\xi(t-q),$$

where $t \in \mathbb{Z}$, $\{\xi_t\}$ is the random sequence constitutes sample of i.i.d. random variables with mean 0 and variance $\sigma^2 < \infty$. The parameters sequence $\{\phi_i(t), i = 1, \ldots, p\}$ and $\{\theta_j(t), j = 1, \ldots, q\}$ are periodic with the same period $T \in \mathbb{N}$, thus for every t holds:

$$\phi_i(t) = \phi_i(t+T), \theta_i(t) = \theta_i(t+T).$$

Stable distribution

Let X be a random variable, it follows the symetric α-stable distribution ($X \sim S\alpha S(\alpha, \sigma)$) with scale parameter σ and stability index α if the characteristic function of X is given by:

$$\phi_X(t) = \mathbb{E}\exp(itX) = \exp(-\sigma^\alpha |t|^\alpha)$$

- There is no closed form of the probability density function;
- For $\alpha = 2$ it is a Gaussian distribution;
- The variance is infinite for $\alpha < 2$;

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Alternative measures of dependency – codifference

Let consider time series $\{X_t\}, t \in \mathbb{Z}$ such that for each t X_t has infinite divisible distribution then the autocodifference for lag k is defined as

$$\mathrm{CD}(X_t, X_{t+k})$$
$$= \log(\mathbb{E}\exp(i(X_t - X_{t+k}))) - \log(\mathbb{E}\exp(iX_t)) - \log(\mathbb{E}\exp(-iX_{t+k}))$$

It can be rewritten using characteristic function

$$\mathrm{CD}(X_t, X_{t+k}) = \log\left(\frac{\phi_{X_t - X_{t+k}}(1)}{\phi_{X_t}(1)\phi_{-X_{t+k}}(1)}\right)$$

Alternative measures of dependency – covariation

Let consider time series $\{X_t\}, t \in \mathbb{Z}$ such that for each t $X_t \sim S\alpha S(\alpha, \sigma)$ and $\alpha \in (1,2]$ then the autocovariation for lag k is defined as:

$$[X_t, X_{t+k}]_\alpha = \int_{S_2} s_1 s_2^{<\alpha-1>} \Gamma(ds),$$

Where S_2 is the unit sphere in $\mathbb{R}^2$, Γ is the spectral measure of random vector (X_t, X_{t+k}) and $z^{<p>} = |p|\text{sign}(z)$.

Alternative measures of dependency – covariation

Using covaration the norm can be defined for $X_t \sim S\alpha S(\alpha, \sigma_{X_t})$:

$$\|X_t\|_\alpha = ([X_t, X_t]_\alpha)^{\frac{1}{\alpha}} = \sigma_{X_t}$$

Let consider time series $\{X_t\}, t \in \mathbb{Z}$ such that for each t $X_t \sim S\alpha S(\alpha, \sigma)$.

Then for all $k \in \mathbb{Z}$ and $1 \leq p < \alpha$:

$$\lambda(X, Y) = \frac{\mathbb{E} X_t X_{t+k}^{<p-1>}}{\mathbb{E}|X_{t+k}|^p} = \frac{[X_t, X_{t+k}]_\alpha}{\|X_{t+k}\|_\alpha^\alpha}$$

Cyclostationary α-stable

Let consider time series $\{X_t\}, t \in \mathbb{Z}$ such that for each t $X_t \sim S\alpha S(\alpha, \sigma)$ with stability index $\alpha > 1$. It is α-stable weakly stationary when $\forall t, k \in \mathbb{Z}$ the following holds:

$$\mathbb{E}X_t = m, \qquad \lambda(X_t, X_{t+k}) = \lambda(X_{t+1}, X_{t+1+k})$$

Let consider time series $\{X_t\}, t \in \mathbb{Z}$ such that for each t $X_t \sim S\alpha S(\alpha, \sigma)$ with stability index $\alpha > 1$. It is α-stable cyclostationary with period $T \in \mathbb{N}$ when $\forall t, s \in \mathbb{Z}$ the following holds:

$$\mathbb{E}X_t = \mathbb{E}X_{t+T}, \qquad \lambda(X_{t+T}, X_{s+T}) = \lambda(X_t, X_s)$$

Cyclic autocovariation

Let consider α-stable cyclostationary with period $T \in \mathbb{N}$ time series $\{X_t\}, t \in \mathbb{Z}$ such that for each t $X_t \sim S\alpha S(\alpha, \sigma)$ with stability index $\alpha > 1$. Then cyclic autocovariation is defined as

$$\lambda_X^{\epsilon}(\tau) = \lim_{N\to\infty} \frac{1}{N+1} \sum_{t=-N/2}^{N/2} \lambda(X_t, X_{t+\tau}) e^{-i2\pi\epsilon t}$$

Where $\epsilon \in A$ and A is countable set, not depending on τ, of possible cycle frequencies ϵ.

Generalized spectra coherence

Let consider α-stable cyclostationary time series $\{X_t\}$ with period $T \in \mathbb{N}, t \in \mathbb{Z}$ such that for each t $X_t \sim S\alpha S(\alpha, \sigma)$ with stability index $\alpha > 1$. Then spectral covariation is defined as

$$\mathrm{SCV}_X(f, \epsilon) = \sum_{\tau=-\infty}^{\infty} \lambda_X^{\epsilon}(\tau) e^{-i2\pi f \tau}.$$

And Generalized spectral coherence is

$$\left|\gamma_X^{\mathrm{CV}}(f, \epsilon)\right|^2 = \lim_{N\to\infty} \frac{\left|\mathrm{SCV}_X^{(N)}(f, \epsilon)\right|^2}{\mathrm{SCV}_X^{(N)}(f + \epsilon/2, 0)\, \mathrm{SCV}_X^{(N)}(f - \epsilon/2, 0)},$$

where

$$\mathrm{SCV}_X^{(N)}(f, \epsilon) = \sum_{\tau=-\infty}^{\infty} \sum_{t=-N/2}^{N/2} \lambda(X_t, X_{t+\tau}) e^{-i2\pi f \tau} e^{-i2\pi \epsilon t}.$$

Integrated SCV

An equivalent to the envelope spectrum can be defined

$$\int_{-1/2}^{1/2} SCV_X(f,\epsilon)df = \lim_{N\to\infty} \frac{1}{N+1} \sum_{t=-N/2}^{N/2} \lambda(X_t, X_t)e^{-i2\pi\epsilon t}.$$

Indeed:

$$\int_{-1/2}^{1/2} SCV_X(f,\epsilon)df = \int_{-1/2}^{1/2} \sum_{\tau=-\infty}^{\infty} \lambda_X^{\epsilon}(\tau)e^{-i2\pi f\tau}df = \sum_{\tau=-\infty}^{\infty} \lambda_X^{\epsilon}(\tau) \int_{-1/2}^{1/2} e^{-i2\pi f\tau}df = \sum_{\tau=-\infty}^{\infty} \lambda_X^{\epsilon}(\tau)\frac{\sin(\pi\tau)}{\pi\tau}.$$

If $\tau \in \mathbb{Z}$, then $(\sin \pi\tau)/(\pi\tau) = \delta_{\tau,0}$ and therefore one has:

$$\int_{-1/2}^{1/2} SCV_X(f,\epsilon)df = \lambda_X^{\epsilon}(0) = \lim_{N\to\infty} \frac{1}{N+1} \sum_{t=-N/2}^{N/2} \lambda(X_t, X_t)e^{-i2\pi\epsilon t}.$$

Simulation study - signal

- cyclostationary process related to faulty bearings (SOI);
- non-cyclic, high amplitude randomly occurring impulsive excitations related to oversized pieces of copper ore falling down into the crusher; it is non-Gaussian noise;
- Gaussian noise - related to general measurement noise.

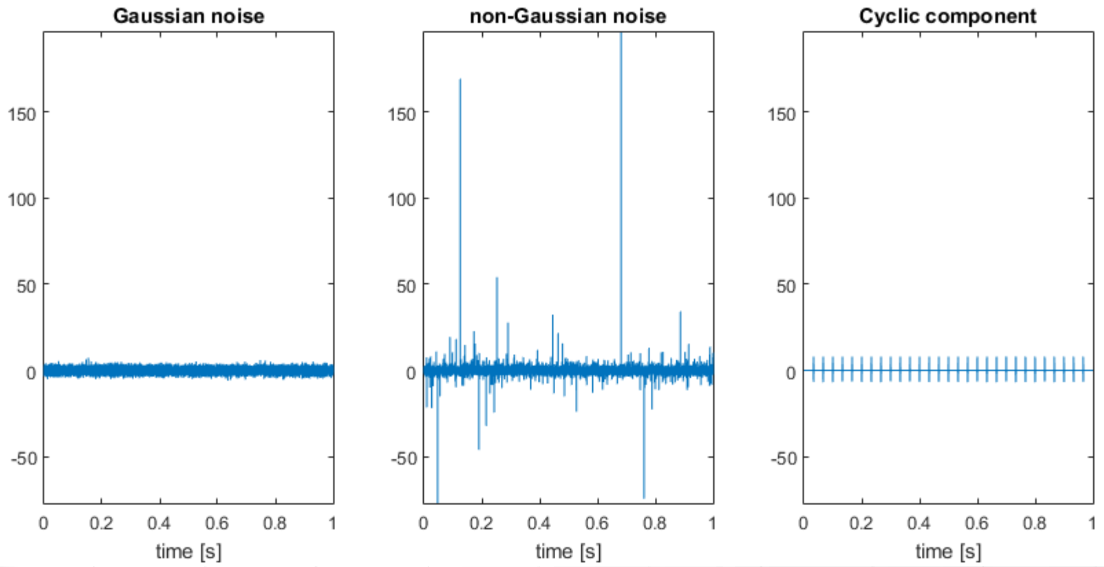
$$s(t) = SOI + noise,$$

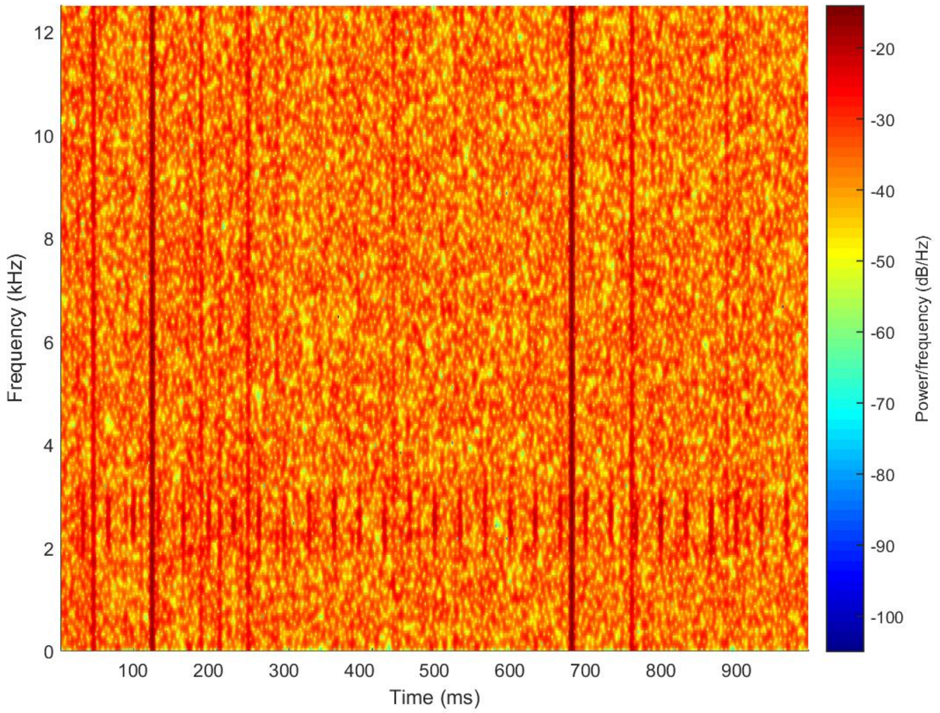
$$noise = non - Gaussian - noise + Gaussian - noise.$$

Simulation study - signal

SOI is a cyclic pulse train with cyclic frequency equal to 30Hz and carrier frequency band equal to 2-3kHz. The amplitude of the cyclic impulses is equal to 8.

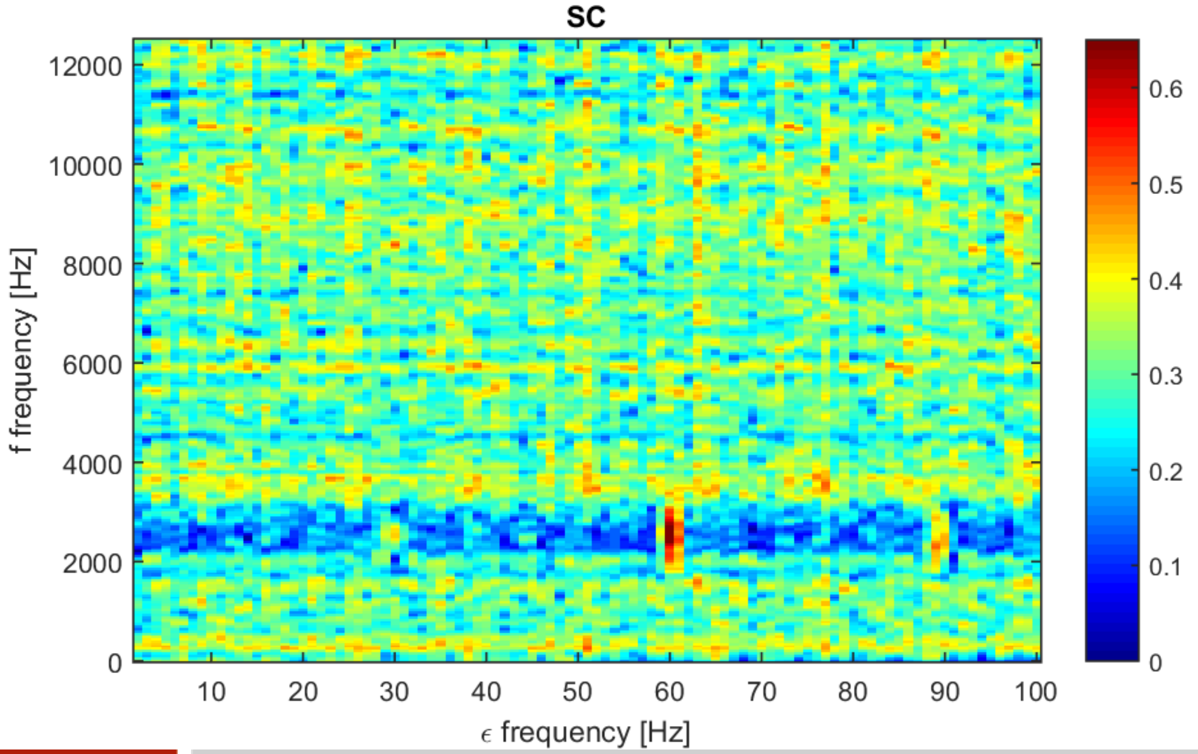
The Gaussian noise has $\mu = 0$ and variance equal to 1.5. Additionally, the non-Gaussian α-stable noise is added with $\alpha = 1.8$ and $\sigma = 0.8$.

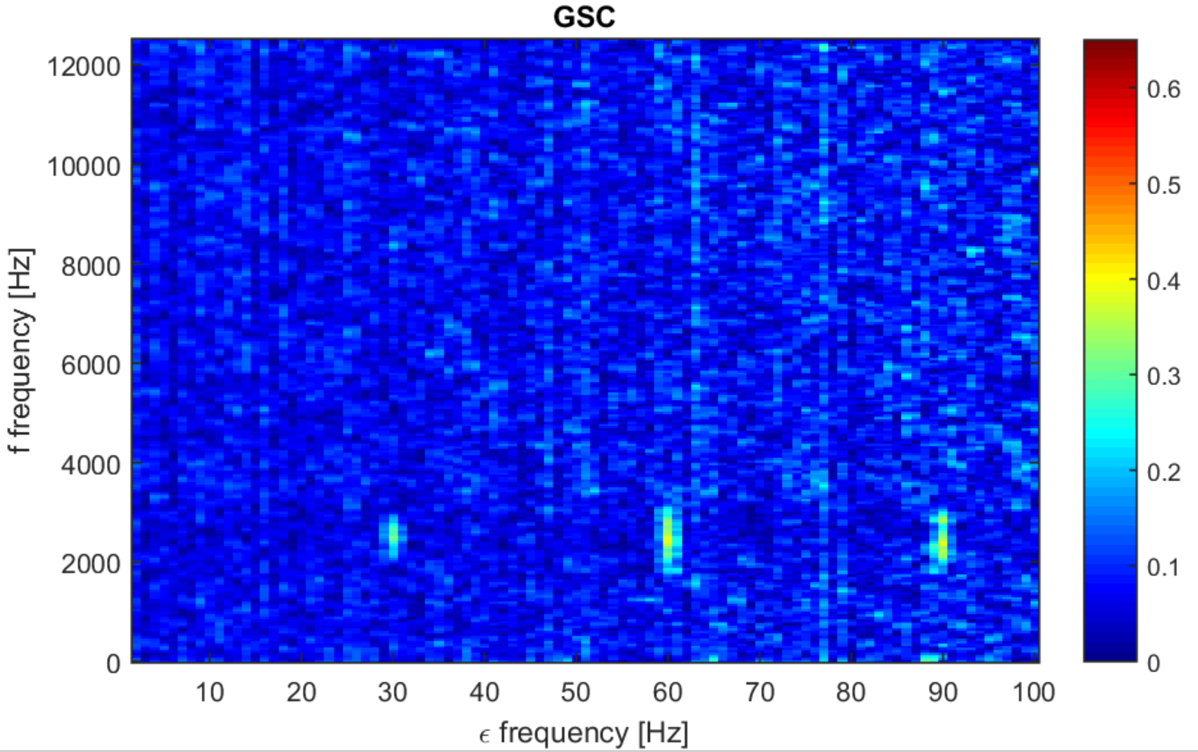




Simulation study – bi-frequency maps

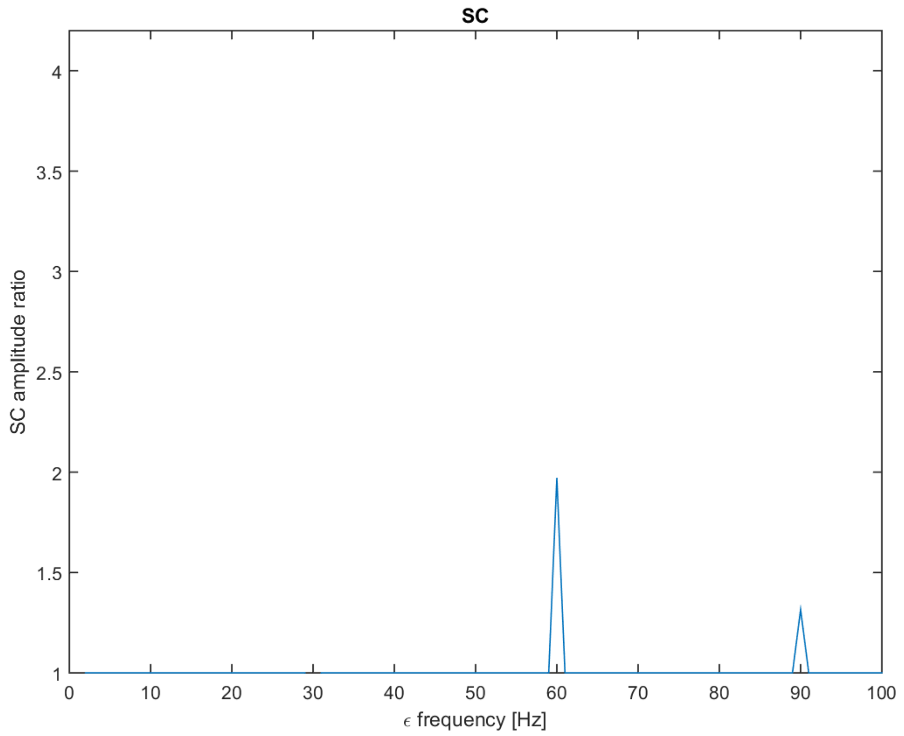
- The cyclic impulses are barely visible and are highly contaminated in SC
- The level of the noise for GSC is significantly smaller than for the spectral coherence.

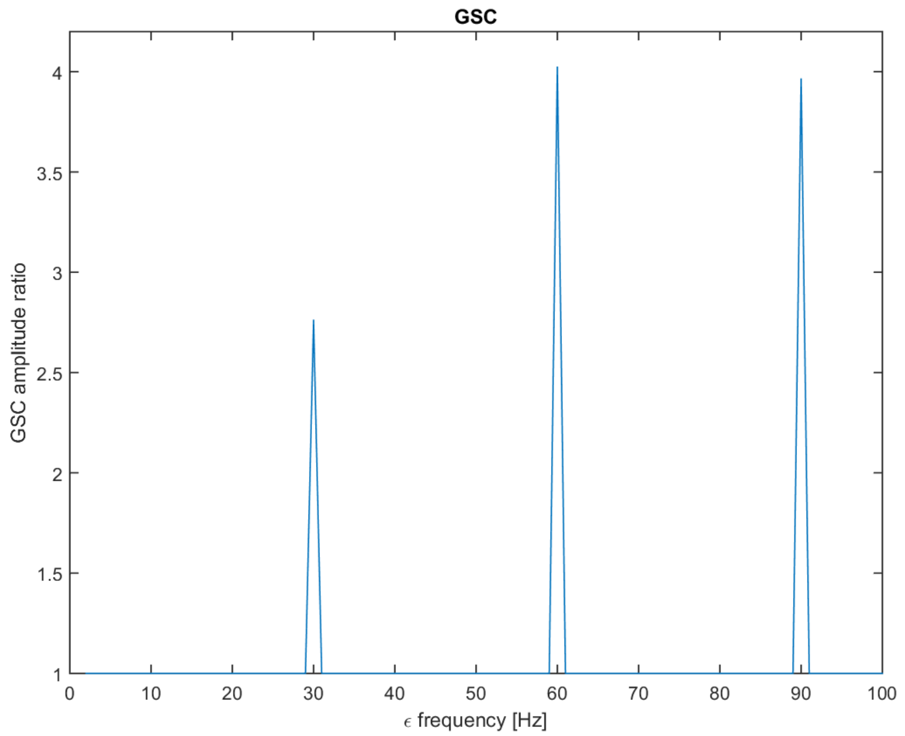




Simulation study – ratio

$$r_{SC}(\epsilon) = \begin{cases} \frac{|\gamma_X(f_c,\epsilon)|^2}{|\overline{\gamma_X}|^2} & \text{if } \epsilon \in \{30, 60, 90\}, \\ 1 & \text{otherwise,} \end{cases}$$





Conclusions

- For the α-stable sequences the covariance is not finite,
- The novel definition of generalised spectral coherence was proposed,
- The covariation is appropriate measure of dependence for α-stable random sequences,
- The generalized spectra coherence is more appropriate for α-stable random sequences.